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Minh Thanh Duong[a], Rosane Ushirobira[b,*]

[a] *Department of Physics, Ho Chi Minh city University of Pedagogy, 280 An Duong Vuong, Ho Chi Minh city, Vietnam*
[b] *Institut de Mathématiques de Bourgogne, Université de Bourgogne, B.P. 47870, F-21078 Dijon Cedex & Non-A team, Inria, France*

Abstract

In this paper, we generalize some results on quadratic Lie algebras to quadratic Lie superalgebras, by applying graded Lie algebras tools. We establish a one-to-one correspondence between non-Abelian quadratic Lie superalgebra structures and nonzero even super-antisymmetric 3-forms satisfying a structure equation. An invariant number of quadratic Lie superalgebras is then defined, called the dup-number. Singular quadratic Lie superalgebras (i.e. those with nonzero dup-number) are studied. We show that their classification follows the classifications of $\mathrm{O}(m)$-adjoint orbits of $\mathfrak{o}(m)$ and $\mathrm{Sp}(2n)$-adjoint orbits of $\mathfrak{sp}(2n)$. An explicit formula for the quadratic dimension of singular quadratic Lie superalgebras is also provided. Finally, we discuss a class of 2-nilpotent quadratic Lie superalgebras associated to a particular super-antisymmetric 3-form.

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0. Introduction

Throughout the paper, the base field is $\mathbb{C}$ and all vector spaces are complex and finite-dimensional. We denote the ring $\mathbb{Z}/2\mathbb{Z}$ by $\mathbb{Z}_2$ as in superalgebra theory.

Let us begin with a $\mathbb{Z}_2$-graded vector space $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$. We say that $\mathfrak{g}$ is a *quadratic* $\mathbb{Z}_2$-graded vector space if it is endowed with a nondegenerate even supersymmetric bilinear form B (that is, B is symmetric on $\mathfrak{g}_{\bar{0}}$, skew-symmetric on $\mathfrak{g}_{\bar{1}}$

*Corresponding author
Email addresses: `thanhdmi@hcmup.edu.vn` (Minh Thanh Duong), `Rosane.Ushirobira@u-bourgogne.fr` (Rosane Ushirobira)

and $B(\mathfrak{g}_{\bar{0}}, \mathfrak{g}_{\bar{1}}) = 0$). If there is a Lie superalgebra structure $[\cdot,\cdot]$ on $\mathfrak{g}$ such that B is invariant,then $\mathfrak{g}$ is called a *quadratic* (or *orthogonal* or *metrised*) Lie superalgebra.

Algebras endowed with an invariant bilinear form appear in many areas of Mathematics and Physics and are a remarkable algebraic object. A structural theory of quadratic Lie algebras, based on the notion of a double extension (a combination of a central extension and a semi-direct product), was introduced by V. Kac [Kac85] in the solvable case and by A. Medina and P. Revoy [MR85] in the general case. Another interesting construction, the T^*-extension, based on the notion of a generalized semi-direct product of a Lie algebra and its dual space was given by M. Bordemann [Bor97] for solvable quadratic Lie algebras. Both notions have been generalized for quadratic Lie superalgebras in papers by H. Benamor and S. Benayadi [BB99] and by I. Bajo, S. Benayadi and M. Bordemann [BBB].

A third approach, based on the concept of super Poisson bracket, was introduced in [PU07], providing several interesting properties of quadratic Lie algebras: the authors consider $(\mathfrak{g}, [\cdot,\cdot], B)$ a non-Abelian quadratic Lie algebra and define a 3-form I on $\mathfrak{g}$ by $I(X,Y,Z) = B([X,Y],Z)$, for all $X,Y,Z \in \mathfrak{g}$. Then I is nonzero and $\{I,I\} = 0$, where $\{\cdot,\cdot\}$ is the super Poisson bracket defined on the ($\mathbb{Z}$-graded) Grassmann algebra $\mathrm{Alt}(\mathfrak{g})$ of $\mathfrak{g}$ by

$$\{\Omega,\Omega'\} = (-1)^{\deg_{\mathbb{Z}}(\Omega)+1} \sum_{j=1}^{n} \iota_{X_j}(\Omega) \wedge \iota_{X_j}(\Omega'),\ \forall\ \Omega,\ \Omega' \in \mathrm{Alt}(\mathfrak{g})$$

with $\{X_1, \dots, X_n\}$ a fixed orthonormal basis of $\mathfrak{g}$. Conversely, given a quadratic vector space $(\mathfrak{g}, B)$ and a nonzero $I \in \mathrm{Alt}^3(\mathfrak{g})$ satisfying $\{I,I\} = 0$, then there is a non-Abelian Lie algebra structure on $\mathfrak{g}$ such that B is invariant.

The element I carries some useful information about corresponding quadratic Lie algebras. For instance, when I is decomposable and nonzero, corresponding quadratic Lie algebras are exhaustively classified [PU07]. In this case, $\dim([\mathfrak{g},\mathfrak{g}]) = 3$ and coadjoint orbits have dimension at most 2. In [DPU12], the authors consider further an invariant that is called the dup-*number* of non-Abelian quadratic Lie algebras and study quadratic Lie algebras with nonzero dup-number. Such quadratic Lie algebras are called *singular*. An unexpected property is that there are many nondegenerate invariant symmetric bilinear forms on a singular quadratic Lie algebra. Though they can be linearly independent, all of them are equivalent in the solvable case. Another remarkable result is that all singular quadratic Lie algebras are classified up to isomorphism by $O(n)$-adjoint orbits of the Lie algebra $\mathfrak{o}(n)$.

The purpose of this paper is to give an interpretation of this last approach for quadratic Lie superalgebras. We combine it with the notion of double extension as it was done for quadratic Lie algebras [DPU12]. Further, we use the notion of generalized double extension. In result, we obtain a rather colorful picture of quadratic Lie superalgebras.

In Section 1, we prove that for a quadratic Lie superalgebra $(\mathfrak{g},B)$ if a 3-form I on $\mathfrak{g}$ is defined by $I(X,Y,Z) = B([X,Y],Z)$, for all $X,Y,Z \in \mathfrak{g}$ then $\{I,I\} = 0$ where $\{.,.\}$ is the super $\mathbb{Z}\times\mathbb{Z}_2$-Poisson bracket given in [MPU09]. Moreover, non-Abelian quadratic Lie superalgebra structures on a quadratic $\mathbb{Z}_2$-graded vector space $(\mathfrak{g},B)$ are in one-to-one correspondence with nonzero even super-antisymmetric trilinear forms I satisfying $\{I,I\} = 0$.

The notion of dup-*number* for a non-Abelian quadratic Lie superalgebra $\mathfrak{g}$ is introduced in Section 2, where the set of singular quadratic Lie superalgebras (that is, with nonzero dup-number) is also studied. We list in Section 3 all non-Abelian reduced quadratic Lie superalgebras with I decomposable. Section 4 details a study of quadratic Lie superalgebras with 2-dimensional even part. We apply the concept of double extension as in [DPU12] with a little change by replacing a quadratic vector space by a symplectic vector space and keeping the other conditions. Then we obtain a classification of quadratic Lie superalgebras with 2-dimensional even part up to isomorphism and isometric isomorphism. This classification follows the classification of $\mathrm{Sp}(2n)$-adjoint orbits of $\mathfrak{sp}(2n)$. We also give a formula for the quadratic dimension of a quadratic Lie superalgebra having a 2-dimensional even part. It indicates that there are many nondegenerate invariant even supersymmetric bilinear forms on a quadratic Lie superalgebra with 2-dimensional even part, but as we shall see, all of them are equivalent.

Section 5 contains results on a singular quadratic Lie superalgebra $(\mathfrak{g},B)$ of type S_1, that is, those with 1-valued dup-number. The first result is that $\mathfrak{g}_{\bar{0}}$ is solvable, so $\mathfrak{g}$ is solvable. Moreover, combining with a result in [DPU12], we prove that two solvable singular quadratic Lie superalgebras are isometrically isomorphic if and only if they are isomorphic and the dup-number is invariant under Lie superalgebra isomorphism. In this way, we obtain a formula for the quadratic dimension of reduced singular quadratic Lie superalgebras of type S_1 with $[\mathfrak{g}_{\bar{1}},\mathfrak{g}_{\bar{1}}] \neq \{0\}$.

In the last Section, we study the structure of a quadratic Lie superalgebra $\mathfrak{g}$ associated to a 3-form $I = J\wedge p$ with $p \in \mathfrak{g}_{\bar{1}}^*$ nonzero and $J \in \mathrm{Alt}^1(\mathfrak{g}_{\bar{0}})\otimes\mathrm{Sym}^1(\mathfrak{g}_{\bar{1}})$ indecomposable, where $\mathrm{Alt}(\mathfrak{g}_{\bar{0}})$ and $\mathrm{Sym}(\mathfrak{g}_{\bar{1}})$ are respectively the algebra of alternating multilinear forms on $\mathfrak{g}_{\bar{0}}$ and symmetric multilinear forms on $\mathfrak{g}_{\bar{1}}$. In this case, $\mathfrak{g}$ is a generalized double extension of a quadratic $\mathbb{Z}_2$-graded vector space and it is 2-nilpotent (see [BBB] for the notion of generalized double extension). The Appendix recalls fundamental results in the classification of $\mathrm{O}(m)$-adjoint orbits of $\mathfrak{o}(m)$ and $\mathrm{Sp}(2n)$-adjoint orbits of $\mathfrak{sp}(2n)$. We further give there the classification of *invertible* orbits, i.e. orbits of isomorphisms in $\mathfrak{o}(m)$ and $\mathfrak{sp}(2n)$. By the Fitting decomposition, we obtain a complete classification in the general case.

Many concepts used in this paper are generalizations of the quadratic Lie algebra case. We do not recall their original definitions here. For more details the reader can refer to [PU07] and [DPU12].

1. Applications of graded Lie algebras to quadratic Lie superalgebras

Let $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ be a $\mathbb{Z}_2$-graded vector space. We call $\mathfrak{g}_{\bar{0}}$ and $\mathfrak{g}_{\bar{1}}$ respectively the *even* and the *odd* part of $\mathfrak{g}$. We begin by reviewing the construction of the super-exterior algebra of the dual space $\mathfrak{g}^*$ of $\mathfrak{g}$. Then we define the super $\mathbb{Z}\times\mathbb{Z}_2$-Poisson bracket on $\mathfrak{g}^*$ (for more details, see [MPU09] and [Sch79]).

1.1. The super-exterior algebra of $\mathfrak{g}^*$

Denote by $\mathrm{Alt}(\mathfrak{g}_{\bar{0}})$ the algebra of alternating multilinear forms on $\mathfrak{g}_{\bar{0}}$ and by $\mathrm{Sym}(\mathfrak{g}_{\bar{1}})$ the algebra of symmetric multilinear forms on $\mathfrak{g}_{\bar{1}}$. Recall that $\mathrm{Alt}(\mathfrak{g}_{\bar{0}})$ is the exterior algebra of $\mathfrak{g}_{\bar{0}}^*$ and $\mathrm{Sym}(\mathfrak{g}_{\bar{1}})$ is the symmetric algebra of $\mathfrak{g}_{\bar{1}}^*$. These algebras are $\mathbb{Z}$-graded algebras. We define a $\mathbb{Z}\times\mathbb{Z}_2$-gradation on $\mathrm{Alt}(\mathfrak{g}_{\bar{0}})$ and on $\mathrm{Sym}(\mathfrak{g}_{\bar{1}})$ by

$$\mathrm{Alt}^{(i,\bar{0})}(\mathfrak{g}_{\bar{0}}) = \mathrm{Alt}^{i}(\mathfrak{g}_{\bar{0}}), \quad \mathrm{Alt}^{(i,\bar{1})}(\mathfrak{g}_{\bar{0}}) = \{0\}$$
$$\text{and} \quad \mathrm{Sym}^{(i,\bar{i})}(\mathfrak{g}_{\bar{1}}) = \mathrm{Sym}^{i}(\mathfrak{g}_{\bar{1}}), \quad \mathrm{Sym}^{(i,\bar{j})}(\mathfrak{g}_{\bar{1}}) = \{0\} \quad \text{if} \quad \bar{i} \neq \bar{j},$$

where $i, j \in \mathbb{Z}$ and $\bar{i}, \bar{j}$ are respectively the residue classes modulo 2 of i and j.

The *super-exterior algebra* of $\mathfrak{g}^*$ is the $\mathbb{Z}\times\mathbb{Z}_2$-graded algebra defined by:

$$\mathcal{E}(\mathfrak{g}) = \mathrm{Alt}(\mathfrak{g}_{\bar{0}}) \underset{\mathbb{Z}\times\mathbb{Z}_2}{\otimes} \mathrm{Sym}(\mathfrak{g}_{\bar{1}})$$

endowed with the *super-exterior product* on $\mathcal{E}(\mathfrak{g})$:

$$(\Omega\otimes F)\wedge(\Omega'\otimes F') = (-1)^{f\omega'}(\Omega\wedge\Omega')\otimes FF',$$

for all $\Omega \in \mathrm{Alt}(\mathfrak{g}_{\bar{0}})$, $\Omega' \in \mathrm{Alt}^{\omega'}(\mathfrak{g}_{\bar{0}})$, $F \in \mathrm{Sym}^{f}(\mathfrak{g}_{\bar{1}})$, $F' \in \mathrm{Sym}(\mathfrak{g}_{\bar{1}})$. Remark that the $\mathbb{Z}\times\mathbb{Z}_2$-gradation on $\mathcal{E}(\mathfrak{g})$ is given by:

$$\text{if } A = \Omega\otimes F \in \mathrm{Alt}^{\omega}(\mathfrak{g}_{\bar{0}})\otimes\mathrm{Sym}^{f}(\mathfrak{g}_{\bar{1}}) \text{ with } \omega,\ f\in\mathbb{Z}, \text{ then } A \in \mathcal{E}^{(\omega+f,\bar{f})}(\mathfrak{g}).$$

So, in terms of the $\mathbb{Z}$-gradations of $\mathrm{Alt}(\mathfrak{g}_{\bar{0}})$ and $\mathrm{Sym}(\mathfrak{g}_{\bar{1}})$, we have:

$$\mathcal{E}^{n}(\mathfrak{g}) = \bigoplus_{m=0}^{n}\left(\mathrm{Alt}^{m}(\mathfrak{g}_{\bar{0}})\otimes\mathrm{Sym}^{n-m}(\mathfrak{g}_{\bar{1}})\right)$$

and in terms of the $\mathbb{Z}_2$-gradations,

$$\mathcal{E}_{\bar{0}}(\mathfrak{g}) = \mathrm{Alt}(\mathfrak{g}_{\bar{0}})\otimes\left(\underset{j\geq 0}{\oplus}\mathrm{Sym}^{2j}(\mathfrak{g}_{\bar{1}})\right) \text{ and } \mathcal{E}_{\bar{1}}(\mathfrak{g}) = \mathrm{Alt}(\mathfrak{g}_{\bar{0}})\otimes\left(\underset{j\geq 0}{\oplus}\mathrm{Sym}^{2j+1}(\mathfrak{g}_{\bar{1}})\right).$$

It is known that the graded vector space $\mathcal{E}(\mathfrak{g})$ endowed with this product is a commutative and associative graded algebra.

Another equivalent construction is given in [BP89]: $\mathcal{E}(\mathfrak{g})$ is the graded algebra of super-antisymmetric multilinear forms on $\mathfrak{g}$. The algebras $\mathrm{Alt}(\mathfrak{g}_{\bar{0}})$ and $\mathrm{Sym}(\mathfrak{g}_{\bar{1}})$ are regarded as subalgebras of $\mathcal{E}(\mathfrak{g})$ by identifying $\Omega := \Omega\otimes 1$, $F := 1\otimes F$, and the tensor product $\Omega\otimes F = (\Omega\otimes 1)\wedge(1\otimes F)$ for all $\Omega\in\mathrm{Alt}(\mathfrak{g}_{\bar{0}})$, $F\in\mathrm{Sym}(\mathfrak{g}_{\bar{1}})$.

1.2. The super $\mathbb{Z}\times\mathbb{Z}_2$-Poisson bracket on $\mathcal{E}(\mathfrak{g})$

Let us assume that the vector space $\mathfrak{g}$ is equipped with a nondegenerate even supersymmetric bilinear form B. That means $B(X,Y) = (-1)^{xy}B(Y,X)$ for all homogeneous $X \in \mathfrak{g}_x$, $Y \in \mathfrak{g}_y$ and $B(\mathfrak{g}_{\bar{0}},\mathfrak{g}_{\bar{1}}) = 0$. In this case, $\dim(\mathfrak{g}_{\bar{1}})$ must be even and $\mathfrak{g}$ is also called a *quadratic $\mathbb{Z}_2$-graded vector space.*

The Poisson bracket on $\mathrm{Sym}(\mathfrak{g}_{\bar{1}})$ and the super Poisson bracket on $\mathrm{Alt}(\mathfrak{g}_{\bar{0}})$ are defined as follows. Let $\mathscr{B} = \{X_1,...,X_n,Y_1,...,Y_n\}$ be a Darboux basis of $\mathfrak{g}_{\bar{1}}$, meaning that $B(X_i,X_j) = B(Y_i,Y_j) = 0$ and $B(X_i,Y_j) = \delta_{ij}$, for all $1 \leq i,j \leq n$. Let $\{p_1,...,p_n,q_1,...,q_n\}$ be its dual basis. Then the algebra $\mathrm{Sym}(\mathfrak{g}_{\bar{1}})$ regarded as the polynomial algebra $\mathbb{C}[p_1,...,p_n,q_1,...,q_n]$ is equipped with the *Poisson bracket*:

$$\{F,G\} = \sum_{i=1}^{n}\left(\frac{\partial F}{\partial p_i}\frac{\partial G}{\partial q_i} - \frac{\partial F}{\partial q_i}\frac{\partial G}{\partial p_i}\right), \text{ for all } F,G \in \mathrm{Sym}(\mathfrak{g}_{\bar{1}}).$$

It is well-known that the algebra $(\mathrm{Sym}(\mathfrak{g}_{\bar{1}}),\{\cdot,\cdot\})$ is a Lie algebra. Now, let $X \in \mathfrak{g}_{\bar{0}}$ and denote by ι_X the derivation of $\mathrm{Alt}(\mathfrak{g}_{\bar{0}})$ defined by:

$$\iota_X(\Omega)(Z_1,\dots,Z_k) = \Omega(X,Z_1,\dots,Z_k),\ \forall\, \Omega \in \mathrm{Alt}^{k+1}(\mathfrak{g}_{\bar{0}}),\ X,Z_1,\dots,Z_k \in \mathfrak{g}_{\bar{0}}\ (k \geq 0),$$

and $\iota_X(1) = 0$. Let $\{Z_1,\dots,Z_m\}$ be a fixed orthonormal basis of $\mathfrak{g}_{\bar{0}}$. Then the *super Poisson bracket* on the algebra $\mathrm{Alt}(\mathfrak{g}_{\bar{0}})$ is defined by (see [PU07] for details):

$$\{\Omega,\Omega'\} = (-1)^{k+1}\sum_{j=1}^{m}\iota_{Z_j}(\Omega)\wedge\iota_{Z_j}(\Omega'),\ \forall\, \Omega \in \mathrm{Alt}^k(\mathfrak{g}_{\bar{0}}),\ \Omega' \in \mathrm{Alt}(\mathfrak{g}_{\bar{0}}).$$

Remark that the definitions above do not depend on the choice of the basis.

Definition 1.1. [MPU09]
The *super $\mathbb{Z}\times\mathbb{Z}_2$-Poisson bracket* on $\mathcal{E}(\mathfrak{g})$ is given by:

$$\{\Omega\otimes F,\Omega'\otimes F'\} = (-1)^{f\omega'}\left(\{\Omega,\Omega'\}\otimes FF' + (\Omega\wedge\Omega')\otimes\{F,F'\}\right),$$

for all $\Omega \in \mathrm{Alt}(\mathfrak{g}_{\bar{0}})$, $\Omega' \in \mathrm{Alt}^{\omega'}(\mathfrak{g}_{\bar{0}})$, $F \in \mathrm{Sym}(\mathfrak{g}_{\bar{1}})^f$, $F' \in \mathrm{Sym}(\mathfrak{g}_{\bar{1}})$.

By a straightforward computation, it is easy to obtain the following result:

Proposition 1.2. *The algebra $\mathcal{E}(\mathfrak{g})$ is a graded Lie algebra with the super $\mathbb{Z}\times\mathbb{Z}_2$-Poisson bracket. More precisely, for all $A \in \mathcal{E}^{(a,b)}(\mathfrak{g})$, $A' \in \mathcal{E}^{(a',b')}(\mathfrak{g})$ and $A'' \in \mathcal{E}^{(a'',b'')}(\mathfrak{g})$:*

1. $\{A',A\} = -(-1)^{aa'+bb'}\{A,A'\}$.
2. $\{\{A,A'\},A''\} = \{A,\{A',A''\}\} - (-1)^{aa'+bb'}\{A',\{A,A''\}\}$.

Moreover, one has $\{A,A'\wedge A''\} = \{A,A'\}\wedge A'' + (-1)^{aa'+bb'}A'\wedge\{A,A''\}$.

1.3. Super-derivations

Denote by $\mathrm{End}(\mathcal{E}(\mathfrak{g}))$ the vector space of endomorphisms of $\mathcal{E}(\mathfrak{g})$. Let $\mathrm{ad}_{\mathrm{P}}(A) := \{A, .\}$, for all $A \in \mathcal{E}(\mathfrak{g})$. Then $\mathrm{ad}_{\mathrm{P}}(A) \in \mathrm{End}(\mathcal{E}(\mathfrak{g}))$ and:

$$\mathrm{ad}_{\mathrm{P}}(\{A, A'\}) = \mathrm{ad}_{\mathrm{P}}(A) \circ \mathrm{ad}_{\mathrm{P}}(A') - (-1)^{aa'+bb'} \mathrm{ad}_{\mathrm{P}}(A') \circ \mathrm{ad}_{\mathrm{P}}(A)$$

for all $A, A' \in \mathcal{E}^{(a',b')}(\mathfrak{g})$. The space $\mathrm{End}(\mathcal{E}(\mathfrak{g}))$ is naturally $\mathbb{Z} \times \mathbb{Z}_2$-graded by

$\deg(F) = (n, d)$, $n \in \mathbb{Z}$, $d \in \mathbb{Z}_2$ if $\deg(F(A)) = (n+a, d+b)$, where $A \in \mathcal{E}^{(a,b)}(\mathfrak{g})$.

We denote by $\mathrm{End}^n_f(\mathcal{E}(\mathfrak{g}))$ the subspace of endomorphisms of degree (n, f) of $\mathrm{End}(\mathcal{E}(\mathfrak{g}))$. It is clear that if $A \in \mathcal{E}^{(a,b)}(\mathfrak{g})$ then $\mathrm{ad}_{\mathrm{P}}(A)$ has degree $(a-2, b)$. Moreover, it is known that $\mathrm{End}(\mathcal{E}(\mathfrak{g}))$ is also a graded Lie algebra, frequently denoted by $\mathfrak{gl}(\mathcal{E}(\mathfrak{g}))$ and equipped with the Lie super-bracket:

$$[F, G] = F \circ G - (-1)^{np+fg} G \circ F, \ \ \forall\, F \in \mathrm{End}^n_f(\mathcal{E}(\mathfrak{g})),\ G \in \mathrm{End}^p_g(\mathcal{E}(\mathfrak{g})).$$

Therefore, by Proposition 1.2, we obtain that ad_{P} is a graded Lie algebra homomorphism from $\mathcal{E}(\mathfrak{g})$ onto $\mathfrak{gl}(\mathcal{E}(\mathfrak{g}))$.

Definition 1.3. A homogeneous endomorphism $D \in \mathfrak{gl}(\mathcal{E}(\mathfrak{g}))$ of degree (n, d) is called a *super-derivation* of degree (n, d) of $\mathcal{E}(\mathfrak{g})$ (for the super-exterior product) if it satisfies the following condition:

$$D(A \wedge A') = D(A) \wedge A' + (-1)^{na+db} A \wedge D(A'),\ \forall\, A \in \mathcal{E}^{(a,b)}(\mathfrak{g}),\ A' \in \mathcal{E}(\mathfrak{g}).$$

Denote by $\mathcal{D}^n_d(\mathcal{E}(\mathfrak{g}))$ the space of super-derivations of degree (n, d) of $\mathcal{E}(\mathfrak{g})$ then there is a $\mathbb{Z} \times \mathbb{Z}_2$-gradation of the space of super-derivations $\mathcal{D}(\mathcal{E}(\mathfrak{g}))$ of $\mathcal{E}(\mathfrak{g})$:

$$\mathcal{D}(\mathcal{E}(\mathfrak{g})) = \bigoplus_{(n,d) \in \mathbb{Z} \times \mathbb{Z}_2} \mathcal{D}^n_d(\mathcal{E}(\mathfrak{g}))$$

and $\mathcal{D}(\mathcal{E}(\mathfrak{g}))$ becomes a graded subalgebra of $\mathfrak{gl}(\mathcal{E}(\mathfrak{g}))$ [NR66]. Moreover, the last formula in Proposition 1.2 affirms that $\mathrm{ad}_{\mathrm{P}}(A) \in \mathcal{D}(\mathcal{E}(\mathfrak{g}))$, for all $A \in \mathcal{E}(\mathfrak{g})$.

Another example of a super-derivation in $\mathcal{D}(\mathcal{E}(\mathfrak{g}))$ is given in [BP89] as follows. Let $X \in \mathfrak{g}_x$ be a homogeneous element in $\mathfrak{g}$ of degree x and define the endomorphism ι_X of $\mathcal{E}(\mathfrak{g})$ by

$$\iota_X(A)(X_1, ..., X_{a-1}) = (-1)^{xb} A(X, X_1, ..., X_{a-1})$$

for all $A \in \mathcal{E}^{(a,b)}(\mathfrak{g})$, $X_1, \ldots, X_{a-1} \in \mathfrak{g}$. Then one has

$$\iota_X(A \wedge A') = \iota_X(A) \wedge A' + (-1)^{-a+xb} A \wedge \iota_X(A')$$

holds for all $A \in \mathcal{E}^{(a,b)}(\mathfrak{g})$, $A' \in \mathcal{E}(\mathfrak{g})$. It means that ι_X is a super- derivation of $\mathcal{E}(\mathfrak{g})$ of degree $(-1, x)$. The proof of the following Lemma is straightforward:

Lemma 1.4. *Let $X_{\bar 0} \in \mathfrak{g}_{\bar 0}$ and $X_{\bar 1} \in \mathfrak{g}_{\bar 1}$. Then, for all $\Omega \otimes F \in \mathrm{Alt}^{\omega}(\mathfrak{g}_{\bar 0}) \otimes \mathrm{Sym}^{f}(\mathfrak{g}_{\bar 1})$:*

$$\iota_{X_{\bar 0}}(\Omega \otimes F) = \iota_{X_{\bar 0}}(\Omega) \otimes F \text{ and } \iota_{X_{\bar 1}}(\Omega \otimes F) = (-1)^{\omega} \Omega \otimes \iota_{X_{\bar 1}}(F).$$

Remark 1.5.

1. If $\Omega \in \mathrm{Alt}^{\omega}(\mathfrak{g}_{\bar 0})$ then $\iota_X(\Omega)(X_1, \ldots, X_{\omega-1}) = \Omega(X, X_1, \ldots, X_{\omega-1})$ for all $X, X_1, \ldots, X_{\omega-1} \in \mathfrak{g}_{\bar 0}$. That coincides with the previous definition of ι_X on $\mathrm{Alt}(\mathfrak{g}_{\bar 0})$.
2. Let X be an element in a fixed Darboux basis of $\mathfrak{g}_{\bar 1}$ and $p \in \mathfrak{g}_{\bar 1}^*$ be its dual form. By the Corollary II.1.52 in [Gié04] one has:

$$\iota_X(p^n)(X^{n-1}) = (-1)^n p^n(X^n) = (-1)^n(-1)^{n(n-1)/2} n!.$$

Moreover, $\dfrac{\partial p^n}{\partial p}(X^{n-1}) = n(p^{n-1})(X^{n-1}) = (-1)^{(n-1)(n-2)/2} n!$ implies that $\iota_X(p^n)(X^{n-1}) = -\dfrac{\partial p^n}{\partial p}(X^{n-1})$. Since each $F \in \mathrm{Sym}^f(\mathfrak{g}_{\bar 1})$ can be regarded as a polynomial in the variable p then $\iota_X(F) = -\dfrac{\partial F}{\partial p}$ for all $F \in \mathrm{Sym}(\mathfrak{g}_{\bar 1})$.

The super-derivations ι_X play an important role in the description of the space $\mathcal{D}(\mathcal{E}(\mathfrak{g}))$ (for details, see [Gié04]). For instance, they can be used to express the super-derivation $\mathrm{ad}_{\mathrm{P}}(A)$ defined above:

Proposition 1.6. *Fix an orthonormal basis $\{X_{\bar 0}^1, \ldots, X_{\bar 0}^m\}$ of $\mathfrak{g}_{\bar 0}$ and a Darboux basis $\mathscr{B} = \{X_{\bar 1}^1, \ldots, X_{\bar 1}^n, Y_{\bar 1}^1, \ldots, Y_{\bar 1}^n\}$ of $\mathfrak{g}_{\bar 1}$. Then the super $\mathbb{Z} \times \mathbb{Z}_2$-Poisson bracket on $\mathcal{E}(\mathfrak{g})$ is given by:*

$$\begin{aligned} \{A, A'\} &= (-1)^{\omega+f+1} \sum_{j=1}^{m} \iota_{X_{\bar 0}^j}(A) \wedge \iota_{X_{\bar 0}^j}(A') \\ &\quad + (-1)^{\omega} \sum_{k=1}^{n} \Big(\iota_{X_{\bar 1}^k}(A) \wedge \iota_{Y_{\bar 1}^k}(A') - \iota_{Y_{\bar 1}^k}(A) \wedge \iota_{X_{\bar 1}^k}(A') \Big) \end{aligned}$$

for all $A \in \mathrm{Alt}^{\omega}(\mathfrak{g}_{\bar 0}) \otimes \mathrm{Sym}^{f}(\mathfrak{g}_{\bar 1})$ and $A' \in \mathcal{E}(\mathfrak{g})$.

Proof. Let $A = \Omega \otimes F \in \mathrm{Alt}^{\omega}(\mathfrak{g}_{\bar 0}) \otimes \mathrm{Sym}^{f}(\mathfrak{g}_{\bar 1})$ and $A' = \Omega' \otimes F' \in \mathrm{Alt}^{\omega'}(\mathfrak{g}_{\bar 0}) \otimes \mathrm{Sym}^{f'}(\mathfrak{g}_{\bar 1})$. The definition of the super Poisson bracket on $\mathrm{Alt}(\mathfrak{g}_{\bar 0})$ and Lemma 1.4 imply that

$$\begin{aligned} \{\Omega, \Omega'\} \otimes FF' &= (-1)^{\omega+1} \sum_{j=1}^{m} \Big(\iota_{X_{\bar 0}^j}(\Omega) \wedge \iota_{X_{\bar 0}^j}(\Omega') \Big) \otimes FF' \\ &= (-1)^{f\omega'+\omega+f+1} \sum_{j=1}^{m} \iota_{X_{\bar 0}^j}(A) \wedge \iota_{X_{\bar 0}^j}(A'). \end{aligned}$$

Let $\{p_1,\dots,p_n,q_1,\dots,q_n\}$ be the dual basis of $\mathscr{B}$. By Remark 1.5 (2) and Lemma 1.4, we obtain

$$
\begin{aligned}
(\Omega\wedge\Omega')\otimes\{F,F'\} &= (\Omega\wedge\Omega')\otimes\sum_{k=1}^{n}\Big(\iota_{X_{\bar 1}^k}(F)\,\iota_{Y_{\bar 1}^k}(F')-\iota_{Y_{\bar 1}^k}(F)\,\iota_{X_{\bar 1}^k}(F')\Big)\\
&= (-1)^{f\omega'+\omega}\sum_{k=1}^{n}\Big(\iota_{X_{\bar 1}^k}(A)\wedge\iota_{Y_{\bar 1}^k}(A')-\iota_{Y_{\bar 1}^k}(A)\wedge\iota_{X_{\bar 1}^k}(A')\Big).
\end{aligned}
$$

The result then follows. □

Since the bilinear form B is nondegenerate and even, then there is an (even) isomorphism ϕ from $\mathfrak{g}$ onto $\mathfrak{g}^*$ defined by $\phi(X)(Y)=B(X,Y)$, for all X, $Y\in\mathfrak{g}$.

Corollary 1.7. *For all $\alpha,\alpha'\in\mathfrak{g}^*$, $A\in\mathcal{E}(\mathfrak{g})$, one has $\{\alpha,A\}=\iota_{\phi^{-1}(\alpha)}(A)$ and $\{\alpha,\alpha'\}=B(\phi^{-1}(\alpha),\phi^{-1}(\alpha'))$.*

In this Section, Proposition 1.6 and Corollary 1.7 are enough for our purpose. But as a consequence of Lemma 6.9 in [PU07], one has a more general result of Proposition 1.6 as follows:

Proposition 1.8. *Let $\{X_{\bar 0}^1,\dots,X_{\bar 0}^m\}$ be a basis of $\mathfrak{g}_{\bar 0}$ and $\{\alpha_1,\dots,\alpha_m\}$ its dual basis. Let $\{Y_{\bar 0}^1,\dots,Y_{\bar 0}^m\}$ be the basis of $\mathfrak{g}_{\bar 0}$ defined by $Y_{\bar 0}^i=\phi^{-1}(\alpha_i)$. Set $\mathscr{B}=\{X_{\bar 1}^1,\dots,X_{\bar 1}^n,Y_{\bar 1}^1,\dots,Y_{\bar 1}^n\}$ be a Darboux basis of $\mathfrak{g}_{\bar 1}$. Then the super $\mathbb{Z}\times\mathbb{Z}_2$-Poisson bracket on $\mathcal{E}(\mathfrak{g})$ is given by*

$$
\begin{aligned}
\{A,A'\} \;=\; & (-1)^{\omega+f+1}\sum_{i,j=1}^{m}B(Y_{\bar 0}^i,Y_{\bar 0}^j)\,\iota_{X_{\bar 0}^i}(A)\wedge\iota_{X_{\bar 0}^j}(A')\\
&+(-1)^{\omega}\sum_{k=1}^{n}\Big(\iota_{X_{\bar 1}^k}(A)\wedge\iota_{Y_{\bar 1}^k}(A')-\iota_{Y_{\bar 1}^k}(A)\wedge\iota_{X_{\bar 1}^k}(A')\Big)
\end{aligned}
$$

for all $A\in\mathrm{Alt}^{\omega}(\mathfrak{g}_{\bar 0})\otimes\mathrm{Sym}^{f}(\mathfrak{g}_{\bar 1})$ and $A'\in\mathcal{E}(\mathfrak{g})$.

1.4. Super-antisymmetric linear maps

Consider the vector space $\mathscr{E}=\bigoplus_{n\in\mathbb{Z}}\mathscr{E}^n$, where $\mathscr{E}^n=\{0\}$ if $n\le -2$, $\mathscr{E}^{-1}=\mathfrak{g}$ and $\mathscr{E}^n$ is the space of super-antisymmetric $n+1$-linear mappings from $\mathfrak{g}^{n+1}$ onto $\mathfrak{g}$. Each of the subspaces $\mathscr{E}^n$ is $\mathbb{Z}_2$-graded then the space $\mathscr{E}$ is $\mathbb{Z}\times\mathbb{Z}_2$-graded by

$$\mathscr{E}=\bigoplus_{f\in\mathbb{Z}_2}^{n\in\mathbb{Z}}\mathscr{E}_f^n.$$

There is a natural isomorphism between the spaces $\mathscr{E}$ and $\mathcal{E}(\mathfrak{g})\otimes\mathfrak{g}$. Moreover, $\mathscr{E}$ is a graded Lie algebra, called the *graded Lie algebra* of $\mathfrak{g}$. It is isomorphic to $\mathcal{D}(\mathcal{E}(\mathfrak{g}))$ by the graded Lie algebra isomorphism D such that if $F=\Omega\otimes X\in\mathscr{E}^n_{\omega+x}$ then $D_F=-(-1)^{x\omega}\Omega\wedge\iota_X\in\mathcal{D}^n_{\omega+x}(\mathcal{E}(\mathfrak{g}))$. For more details on the Lemma below, see for instance, [BP89] and [Gié04].

Lemma 1.9. *Fix $F\in\mathscr{E}^1_{\overline{0}}$, denote by $d=D_F$ and define the product $[X,Y]=F(X,Y)$, for all $X,Y\in\mathfrak{g}$. Then one has*

1. *$d(\phi)(X,Y)=-\phi([X,Y])$, for all $X,Y\in\mathfrak{g}$, $\phi\in\mathfrak{g}^*$.*
2. *The product $[\ ,\ ]$ becomes a Lie super-bracket if and only if $d^2=0$. In this case, d is called a* super-exterior differential *of $\mathcal{E}(\mathfrak{g})$.*

1.5. Quadratic Lie superalgebras

The construction of graded Lie algebras and the super $\mathbb{Z}\times\mathbb{Z}_2$-Poisson bracket above can be applied to the theory of quadratic Lie superalgebras. This later is regarded as a graded version of the quadratic Lie algebra case and we obtain then similar results.

Definition 1.10. A *quadratic* Lie superalgebra $(\mathfrak{g},B)$ is a $\mathbb{Z}_2$-graded vector space $\mathfrak{g}$ equipped with a nondegenerate even supersymmetric bilinear form B and a Lie superalgebra structure such that B is invariant, i.e. $B([X,Y],Z)=B(X,[Y,Z])$, for all X, Y, $Z\in\mathfrak{g}$.

Proposition 1.11. *Let $(\mathfrak{g},B)$ be a quadratic Lie superalgebra and define a trilinear form I on $\mathfrak{g}$ by $I(X,Y,Z)=B([X,Y],Z)$ for all $X,Y,Z\in\mathfrak{g}$. Then one has*

1. $I\in\mathcal{E}^{(3,\overline{0})}(\mathfrak{g})=\mathrm{Alt}^3(\mathfrak{g}_{\overline{0}})\oplus\left(\mathrm{Alt}^1(\mathfrak{g}_{\overline{0}})\otimes\mathrm{Sym}^2(\mathfrak{g}_{\overline{1}})\right).$
2. *$d=-\operatorname{ad}_{\mathrm{P}}(I)$ and $\{I,I\}=0$.*

Proof. The assertion (1) follows clearly the properties of B. Note that $B([\mathfrak{g}_{\overline{0}},\mathfrak{g}_{\overline{0}}],\mathfrak{g}_{\overline{1}})=B([\mathfrak{g}_{\overline{1}},\mathfrak{g}_{\overline{1}}],\mathfrak{g}_{\overline{1}})=0$.

For (2), fix $\{X^1_{\overline{0}},\dots,X^m_{\overline{0}}\}$ an orthonormal basis of $\mathfrak{g}_{\overline{0}}$ and $\{X^1_{\overline{1}},\dots,X^n_{\overline{1}},Y^1_{\overline{1}},\dots,Y^n_{\overline{1}}\}$ a Darboux basis of $\mathfrak{g}_{\overline{1}}$. Let $\{\alpha_1,\dots,\alpha_m\}$ and $\{\beta_1,\dots,\beta_n,\gamma_1,\dots,\gamma_n\}$ be their dual basis, respectively. For all $X,Y\in\mathfrak{g}$, $i=1,...,m$, $l=1,...,n$ we have:

$$\begin{aligned}
&\operatorname{ad}_{\mathrm{P}}(I)(\alpha_i)(X,Y)\\
&=\left(\sum_{j=1}^m\iota_{X^j_{\overline{0}}}(I)\wedge\iota_{X^j_{\overline{0}}}(\alpha_i)-\sum_{k=1}^n\left(\iota_{X^k_{\overline{1}}}(I)\wedge\iota_{Y^k_{\overline{1}}}(\alpha_i)-\iota_{Y^k_{\overline{1}}}(I)\wedge\iota_{X^k_{\overline{1}}}(\alpha_i)\right)\right)(X,Y)\\
&=\left(\iota_{X^i_{\overline{0}}}(I)\wedge\iota_{X^i_{\overline{0}}}(\alpha_i)\right)(X,Y)=B(X^i_{\overline{0}},[X,Y])=\alpha_i([X,Y])=-d(\alpha_i)(X,Y).
\end{aligned}$$

Similarly, $\mathrm{ad}_{\mathrm{P}}(I)(\beta_l) = -d(\beta_l)$ and $\mathrm{ad}_{\mathrm{P}}(I)(\gamma_l) = -d(\gamma_l)$ for $1 \leq l \leq n$. Therefore, $d = -\mathrm{ad}_{\mathrm{P}}(I)$.

Moreover, $\mathrm{ad}_{\mathrm{P}}(\{I,I\}) = [\mathrm{ad}_{\mathrm{P}}(I), \mathrm{ad}_{\mathrm{P}}(I)] = [d,d] = 2d^2 = 0$. Then for all $1 \leq i \leq m$, $1 \leq j,k \leq n$ one has $\{\alpha_i, \{I,I\}\} = \{\beta_j, \{I,I\}\} = \{\gamma_k, \{I,I\}\} = 0$. Those imply $\iota_X(\{I,I\}) = 0$ for all $X \in \mathfrak{g}$ and hence, we obtain $\{I,I\} = 0$. □

Conversely, let $\mathfrak{g}$ be a quadratic $\mathbb{Z}_2$-graded vector space equipped with a bilinear form B and I be an element in $\mathcal{E}^{(3,\bar{0})}(\mathfrak{g})$. Define $d = -\mathrm{ad}_{\mathrm{P}}(I)$ then $d \in \mathcal{D}^1_{\bar{0}}(\mathcal{E}(\mathfrak{g}))$. Therefore, $d^2 = 0$ if and only if $\{I,I\} = 0$. Let F be the structure in $\mathfrak{g}$ corresponding to d by the isomorphism D in Lemma 1.9.

Proposition 1.12. *F becomes a Lie superalgebra structure if and only if $\{I,I\} = 0$. In this case, with the notation $[X,Y] := F(X,Y)$ one has $I(X,Y,Z) = B([X,Y],Z)$ for all $X,Y,Z \in \mathfrak{g}$. Moreover, the bilinear form B is invariant.*

Proof. We need to prove that if F is a Lie superalgebra structure then $I(X,Y,Z) = B([X,Y],Z)$, for all $X,Y,Z \in \mathfrak{g}$. Indeed, let $\{X^1_{\bar{0}}, \ldots, X^m_{\bar{0}}\}$ be an orthonormal basis of $\mathfrak{g}_{\bar{0}}$ and $\{X^1_{\bar{1}}, \ldots, X^n_{\bar{1}}, Y^1_{\bar{1}}, \ldots, Y^n_{\bar{1}}\}$ be a Darboux basis of $\mathfrak{g}_{\bar{1}}$ then one has

$$d = -\mathrm{ad}_{\mathrm{P}}(I) = -\sum_{j=1}^{m} \iota_{X^j_{\bar{0}}}(I) \wedge \iota_{X^j_{\bar{0}}} + \sum_{k=1}^{n} \iota_{X^k_{\bar{1}}}(I) \wedge \iota_{Y^k_{\bar{1}}} - \sum_{k=1}^{n} \iota_{Y^k_{\bar{1}}}(I) \wedge \iota_{X^k_{\bar{1}}}.$$

It implies that $F = \sum_{j=1}^{m} \iota_{X^j_{\bar{0}}}(I) \otimes X^j_{\bar{0}} + \sum_{k=1}^{n} \iota_{X^k_{\bar{1}}}(I) \otimes Y^k_{\bar{1}} - \sum_{k=1}^{n} \iota_{Y^k_{\bar{1}}}(I) \otimes X^k_{\bar{1}}$. Therefore, for all i we obtain $B([X,Y],X^i_{\bar{0}}) = \iota_{X^i_{\bar{0}}}(I)(X,Y) = I(X^i_{\bar{0}},X,Y) = I(X,Y,X^i_{\bar{0}})$.

By a similar computation, $B([X,Y],X^i_{\bar{1}}) = I(X,Y,X^i_{\bar{1}})$ and $B([X,Y],Y^i_{\bar{1}}) = I(X,Y,Y^i_{\bar{1}})$. These show that $I(X,Y,Z) = B([X,Y],Z)$, for all $X,Y,Z \in \mathfrak{g}$. Since I is super-antisymmetric and B is supersymmetric, then one can show B invariant. □

The two previous propositions show that on a quadratic $\mathbb{Z}_2$-graded vector space $(\mathfrak{g},B)$, quadratic Lie superalgebra structures with the same B are in one-to-one correspondence with elements $I \in \mathcal{E}^{(3,\bar{0})}(\mathfrak{g})$ satisfying $\{I,I\} = 0$ and such that the super-exterior differential of $\mathcal{E}(\mathfrak{g})$ is $d = -\mathrm{ad}_{\mathrm{P}}(I)$. This correspondence provides an approach to the theory of quadratic Lie superalgebras through I.

Definition 1.13. For a quadratic Lie superalgebra $(\mathfrak{g},B)$, the element I defined above is also an invariant of $\mathfrak{g}$ since $\mathscr{L}_X(I) = 0$, for all $X \in \mathfrak{g}$ where $\mathscr{L}_X = D(\mathrm{ad}_{\mathfrak{g}}(X))$ is the Lie super-derivation of $\mathfrak{g}$. We call I the *associated invariant* of $\mathfrak{g}$.

The following Lemma is a simple, yet interesting result.

Lemma 1.14. *Let $(\mathfrak{g},B)$ be a quadratic Lie superalgebra and I be its associated invariant. Then $\iota_X(I) = 0$ if and only if $X \in \mathcal{Z}(\mathfrak{g})$.*

Definition 1.15. We say that two quadratic Lie superalgebras $(\mathfrak{g},B)$ and $(\mathfrak{g}',B')$ are *isometrically isomorphic* (or *i-isomorphic*) if there exists a Lie superalgebra isomorphism $A:\mathfrak{g}\to\mathfrak{g}'$ satisfying $B'(A(X),A(Y))=B(X,Y)$ for all $X,Y\in\mathfrak{g}$. In this case, we write $\mathfrak{g}\overset{\mathrm{i}}{\simeq}\mathfrak{g}'$.

Note that two isomorphic quadratic Lie superalgebras $(\mathfrak{g},B)$ and $(\mathfrak{g}',B')$ are not necessarily i-isomorphic by the example below:

Example 1.16. Let $\mathfrak{g}=\mathfrak{osp}(1,2)$ and B its Killing form. Recall that $\mathfrak{g}_{\bar{0}}=\mathfrak{o}(3)$. Consider another bilinear form $B'=\lambda B$, $\lambda\in\mathbb{C}$, $\lambda\neq 0$. Then $(\mathfrak{g},B)$ and $(\mathfrak{g},\lambda B)$ can not be i-isomorphic if $\lambda\neq 1$ since $(\mathfrak{g}_{\overline{0}},B)$ and $(\mathfrak{g}_{\overline{0}},\lambda B)$ are not i-isomorphic.

2. The dup-number of quadratic Lie superalgebras

Let $(\mathfrak{g},B)$ be a quadratic Lie superalgebra and I be its associated invariant. By Proposition 1.11 we have a decomposition $I=I_0+I_1$ where $I_0\in\mathrm{Alt}^3(\mathfrak{g}_{\bar{0}})$ and $I_1\in\mathrm{Alt}^1(\mathfrak{g}_{\bar{0}})\otimes\mathrm{Sym}^2(\mathfrak{g}_{\bar{1}})$. Since $\{I,I\}=0$, then $\{I_0,I_0\}=0$. So $\mathfrak{g}_{\bar{0}}$ is a quadratic Lie algebra with the associated 3-form I_0, a rather obvious result. It is easy to see that $\mathfrak{g}_{\bar{0}}$ is Abelian (resp. $[\mathfrak{g}_{\bar{1}},\mathfrak{g}_{\bar{1}}]=\{0\}$) if and only if $I_0=0$ (resp. $I_1=0$). These cases will be fully studied in the sequel. Define the following subspaces of $\mathfrak{g}^*$:

$$\begin{aligned}\mathcal{V}_I &= \{\alpha\in\mathfrak{g}^* \mid \alpha\wedge I=0\},\\ \mathcal{V}_{I_0} &= \{\alpha\in\mathfrak{g}_{\bar{0}}^* \mid \alpha\wedge I_0=0\} \text{ and } \mathcal{V}_{I_1}=\{\alpha\in\mathfrak{g}_{\bar{0}}^* \mid \alpha\wedge I_1=0\}.\end{aligned}$$

Lemma 2.1. *Let $\mathfrak{g}$ be a non-Abelian quadratic Lie superalgebra then one has* $\dim(\mathcal{V}_I)\in\{0,1,3\}$. *Moreover,* $\dim(\mathcal{V}_I)=3$ *if and only if $I_1=0$, $\mathfrak{g}_{\bar{0}}$ is non-Abelian and I_0 is decomposable in* $\mathrm{Alt}^3(\mathfrak{g}_{\bar{0}})$.

Proof. Assume $\alpha=\alpha_0+\alpha_1\in\mathfrak{g}_{\bar{0}}^*\oplus\mathfrak{g}_{\bar{1}}^*$. It is easy to see that $\alpha\wedge I=0$ if and only if $\alpha_1=0$ and $\alpha_0\wedge I_0=\alpha_0\wedge I_1=0$. It means that $\mathcal{V}_I=\mathcal{V}_{I_0}\cap\mathcal{V}_{I_1}$. If $I_0\neq 0$ then $\dim(\mathcal{V}_{I_0})\in\{0,1,3\}$ and if $I_1\neq 0$ then $\dim(\mathcal{V}_{I_1})\in\{0,1\}$. Therefore, $\dim(\mathcal{V}_I)\in\{0,1,3\}$, and $\dim(\mathcal{V}_I)=3$ if and only if $I_1=0$ and $\dim(\mathcal{V}_{I_0})=3$. $\square$

The previous Lemma allows us to introduce the notion of dup-*number* for quadratic Lie superalgebras as we did for quadratic Lie

Definition 2.2. Let $(\mathfrak{g},B)$ be a non-Abelian quadratic Lie superalgebra and I be its associated invariant. The dup-*number* $\mathrm{dup}(\mathfrak{g})$ is defined by $\mathrm{dup}(\mathfrak{g})=\dim(\mathcal{V}_I)$.

Given a subspace W of $\mathfrak{g}$, if W is *nondegenerate* (with respect to B), i.e. the restriction of B on $W\times W$ is nondegenerate, then the orthogonal subspace $W^\perp$ of W is also nondegenerate. In this case, we use the notation $\mathfrak{g}=W\overset{\perp}{\oplus}W^\perp$.

The decomposition result below is a generalization of the quadratic Lie algebra case. Its proof can be found in [PU07] and [DPU12].

Proposition 2.3. *Let $(\mathfrak{g},B)$ be a non-Abelian quadratic Lie superalgebra. Then there are a central ideal $\mathfrak{z}$ and an ideal $\mathfrak{l}\neq\{0\}$ such that:*

1. $\mathfrak{g}=\mathfrak{z}\overset{\perp}{\oplus}\mathfrak{l}$ *where $(\mathfrak{z},B|_{\mathfrak{z}\times\mathfrak{z}})$ and $(\mathfrak{l},B|_{\mathfrak{l}\times\mathfrak{l}})$ are quadratic Lie superalgebras. Moreover, $\mathfrak{l}$ is non-Abelian and its center $\mathcal{Z}(\mathfrak{l})$ is totally isotropic, i.e. $\mathcal{Z}(\mathfrak{l})\subset[\mathfrak{l},\mathfrak{l}]$.*
2. *Let $\mathfrak{g}'$ be a quadratic Lie superalgebra and $A:\mathfrak{g}\to\mathfrak{g}'$ be a Lie superalgebra isomorphism. Then $\mathfrak{g}'=\mathfrak{z}'\overset{\perp}{\oplus}\mathfrak{l}'$ where $\mathfrak{z}'=A(\mathfrak{z})$ is central, $\mathfrak{l}'=A(\mathfrak{z})^{\perp}$, $\mathcal{Z}(\mathfrak{l}')$ is totally isotropic and $\mathfrak{l}$ and $\mathfrak{l}'$ are isomorphic. Moreover if A is an i-isomorphism, then $\mathfrak{l}$ and $\mathfrak{l}'$ are i-isomorphic.*

The Lemma below shows that the previous decomposition has a good behavior with respect to the dup-number. Its proof is rather obvious.

Lemma 2.4. *Let $\mathfrak{g}$ be a non-Abelian quadratic Lie superalgebra. Write $\mathfrak{g}=\mathfrak{z}\overset{\perp}{\oplus}\mathfrak{l}$ as in Proposition 2.3 then* $\mathrm{dup}(\mathfrak{g})=\mathrm{dup}(\mathfrak{l})$.

Clearly, $\mathfrak{z}=\{0\}$ if and only if $\mathcal{Z}(\mathfrak{g})$ is totally isotropic. By the above Lemma, it is enough to restrict our study on the dup-number of non- Abelian quadratic Lie superalgebras with totally isotropic center.

Definition 2.5. A quadratic Lie superalgebra $\mathfrak{g}$ is *reduced* if it satisfies: $\mathfrak{g}\neq\{0\}$ and $\mathcal{Z}(\mathfrak{g})$ is totally isotropic.

Definition 2.6. Let $\mathfrak{g}$ be a non-Abelian quadratic Lie superalgebra. We say that:

1. $\mathfrak{g}$ is an *ordinary* quadratic Lie superalgebra if $\mathrm{dup}(\mathfrak{g})=0$.
2. $\mathfrak{g}$ is a *singular* quadratic Lie superalgebra if $\mathrm{dup}(\mathfrak{g})\geq 1$.
   (i) $\mathfrak{g}$ is a *singular* quadratic Lie superalgebra of *type* S_1 if $\mathrm{dup}(\mathfrak{g})=1$.
   (ii) $\mathfrak{g}$ is a *singular* quadratic Lie superalgebra of *type* S_3 if $\mathrm{dup}(\mathfrak{g})=3$.

By Lemma 2.1, if $\mathfrak{g}$ is singular of type S_3 then $I=I_0$ is decomposable in $\mathrm{Alt}^3(\mathfrak{g}_{\bar{0}})$. One has $I(\mathfrak{g}_{\bar{0}},\mathfrak{g}_{\bar{1}},\mathfrak{g}_{\bar{1}})=B([\mathfrak{g}_{\bar{0}},\mathfrak{g}_{\bar{1}}],\mathfrak{g}_{\bar{1}})=0$ so $[\mathfrak{g}_{\bar{1}},\mathfrak{g}]=\{0\}$ since B is non-degenerate. Hence in this case, $\mathfrak{g}_{\bar{1}}$ is a central ideal, $\mathfrak{g}_{\bar{0}}$ is a singular quadratic Lie algebra of type S_3 and then the classification is known in [PU07]. Therefore, we are mainly interested in singular quadratic Lie superalgebras of type S_1.

Next, we give other simple properties of singular quadratic Lie superalgebras:

Proposition 2.7. *Let $(\mathfrak{g},B)$ be a singular quadratic Lie superalgebra. If $\mathfrak{g}_{\bar{0}}$ is non-Abelian then $\mathfrak{g}_{\bar{0}}$ is a singular quadratic Lie algebra.*

Proof. The result is obvious since $\mathcal{V}_I=\mathcal{V}_{I_0}\cap\mathcal{V}_{I_1}$. □

Given $(\mathfrak{g},B)$ a singular quadratic Lie superalgebra of type S_1. Fix $\alpha \in \mathcal{V}_I$ and choose $\Omega_0 \in \mathrm{Alt}^2(\mathfrak{g}_{\bar{0}})$, $\Omega_1 \in \mathrm{Sym}^2(\mathfrak{g}_{\bar{1}})$ such that $I = \alpha \wedge \Omega_0 + \alpha \otimes \Omega_1$. Then

$$\{I,I\} = \{\alpha \wedge \Omega_0, \alpha \wedge \Omega_0\} + 2\{\alpha \wedge \Omega_0, \alpha\} \otimes \Omega_1 + \{\alpha,\alpha\} \otimes \Omega_1\Omega_1.$$

By the equality $\{I,I\} = 0$, one has $\{\alpha \wedge \Omega_0, \alpha \wedge \Omega_0\} = 0$, $\{\alpha,\alpha\} = 0$ and $\{\alpha, \alpha \wedge \Omega_0\} = 0$. These imply that $\{\alpha, I\} = 0$. Hence, if we set $X_0 = \phi^{-1}(\alpha)$ then $X_0 \in \mathcal{Z}(\mathfrak{g})$ and $B(X_0,X_0) = 0$ (Corollary 1.7 and Lemma 1.14).

Proposition 2.8. *Let $(\mathfrak{g},B)$ be a singular quadratic Lie superalgebra. If $\mathfrak{g}$ is reduced then $\mathfrak{g}_{\bar{0}}$ is reduced.*

Proof. As above, if $\mathfrak{g}$ is singular of type S_3 then $\mathfrak{g}_{\bar{1}}$ is central. By $\mathfrak{g}$ reduced and $\mathfrak{g}_{\bar{1}}$ nondegenerate, $\mathfrak{g}_{\bar{1}}$ must be zero and then the result follows.

If $\mathfrak{g}$ is singular of type S_1. Assume that $\mathfrak{g}_{\bar{0}}$ is not reduced, i.e. $\mathfrak{g}_{\bar{0}} = \mathfrak{z} \overset{\perp}{\oplus} \mathfrak{l}$ where $\mathfrak{z}$ is a nontrivial central ideal of $\mathfrak{g}_{\bar{0}}$, there is $X \in \mathfrak{z}$ such that $B(X,X) = 1$. Since $\mathfrak{g}$ is singular of type S_1 then $\mathfrak{g}_{\bar{0}}$ is also singular. Hence, the element X_0 defined as above must be in $\mathfrak{l}$ and $I_0 = \alpha \wedge \Omega_0 \in \mathrm{Alt}^3(\mathfrak{l})$ (see [DPU12] for details). We also have $B(X,X_0) = 0$.

Let $\beta = \phi(X)$ so $\iota_X(I) = \{\beta, I\} = \{\beta, \alpha \wedge \Omega_0 + \alpha \otimes \Omega_1\} = 0$. That means $X \in \mathcal{Z}(\mathfrak{g})$. This is a contradiction since $\mathfrak{g}$ is reduced. Hence $\mathfrak{g}_{\bar{0}}$ must be reduced. $\square$

Definition 2.9. A quadratic Lie superalgebra $\mathfrak{g}$ is *indecomposable* if $\mathfrak{g} = \mathfrak{g}_1 \overset{\perp}{\oplus} \mathfrak{g}_2$, with $\mathfrak{g}_1$ and $\mathfrak{g}_2$ ideals of $\mathfrak{g}$, then $\mathfrak{g}_1$ or $\mathfrak{g}_2 = \{0\}$.

The proof of the following Proposition is a direct translation from a result in [DPU12] to the Lie superalgebra case.

Proposition 2.10. *Let $\mathfrak{g}$, g_1 and $\mathfrak{g}_2$ be quadratic Lie superalgebras.*

1. *Let $\mathfrak{g}$ be singular. Then $\mathfrak{g}$ is reduced if and only if $\mathfrak{g}$ is indecomposable.*
2. *Let $\mathfrak{g}_1$ and $\mathfrak{g}_2$ be non-Abelian. Then $\mathfrak{g}_1 \overset{\perp}{\oplus} \mathfrak{g}_2$ is an ordinary quadratic Lie superalgebra.*

3. Elementary quadratic Lie superalgebras

In this Section, we consider the first example of singular quadratic Lie superalgebras: elementary quadratic Lie superalgebras. Their definition is as follows.

Definition 3.1. Let $\mathfrak{g}$ be a quadratic Lie superalgebra and I be its associated invariant. We say $\mathfrak{g}$ an *elementary* quadratic Lie superalgebra if I is decomposable.

Keep notations as in Section 2. If $I = I_0 + I_1$ is decomposable, where $I_0 \in \mathrm{Alt}^3(\mathfrak{g}_{\bar{0}})$ and $I_1 \in \mathrm{Alt}^1(\mathfrak{g}_{\bar{0}}) \otimes \mathrm{Sym}^2(\mathfrak{g}_{\bar{1}})$ then it is obvious that I_0 or I_1 is zero. The case $I_1 = 0$ corresponds to singular quadratic Lie superalgebras of type S_3 and then there is nothing to do. Now we assume I is a nonzero decomposable element in $\mathrm{Alt}^1(\mathfrak{g}_{\bar{0}}) \otimes \mathrm{Sym}^2(\mathfrak{g}_{\bar{1}})$ then I can be written by $I = \alpha \otimes pq$, where $\alpha \in \mathfrak{g}_{\bar{0}}^*$ and $p, q \in \mathfrak{g}_{\bar{1}}^*$. It is clear that $\mathfrak{g}$ is singular of type S_1.

Lemma 3.2. *Let $\mathfrak{g}$ be a reduced elementary quadratic Lie superalgebra having $I = \alpha \otimes pq$ where $\alpha \in \mathfrak{g}_{\bar{0}}^*$ and $p, q \in \mathfrak{g}_{\bar{1}}^*$. Set $X_{\bar{0}} = \phi^{-1}(\alpha)$ then one has:*

1. $\dim(\mathfrak{g}_{\bar{0}}) = 2$ *and* $\mathfrak{g}_{\bar{0}} \cap \mathcal{Z}(\mathfrak{g}) = \mathbb{C}X_{\bar{0}}$.
2. *Let $X_{\bar{1}} = \phi^{-1}(p)$, $Y_{\bar{1}} = \phi^{-1}(q)$ and $U = \mathrm{span}\{X_{\bar{1}}, Y_{\bar{1}}\}$ then*
   (i) $\dim(\mathfrak{g}_{\bar{1}}) = 2$ *if* $\dim(U) = 1$ *or U is nondegenerate.*
   (ii) $\dim(\mathfrak{g}_{\bar{1}}) = 4$ *if U is totally isotropic.*

Proof.

1. Let $\beta \in \mathfrak{g}_{\bar{0}}^*$. It is easy to see that $\{\beta, \alpha\} = 0$ if and only if $\{\beta, I\} = 0$, equivalently $\phi^{-1}(\beta) \in \mathcal{Z}(\mathfrak{g})$. Then $(\phi^{-1}(\alpha))^\perp \cap \mathfrak{g}_{\bar{0}} \subset \mathcal{Z}(\mathfrak{g})$. It means that $\dim(\mathfrak{g}_{\bar{0}}) \leq 2$ since $\mathfrak{g}$ is reduced (see [Bou59]). Moreover, $X_{\bar{0}} = \phi^{-1}(\alpha)$ is isotropic then $\dim(\mathfrak{g}_{\bar{0}}) = 2$. If $\dim(\mathfrak{g}_{\bar{0}} \cap \mathcal{Z}(\mathfrak{g})) = 2$ then $\mathfrak{g}_{\bar{0}} \subset \mathcal{Z}(\mathfrak{g})$. Since B is invariant we obtain $\mathfrak{g}$ Abelian (a contradiction). Therefore, $\mathfrak{g}_{\bar{0}} \cap \mathcal{Z}(\mathfrak{g}) = \mathbb{C}X_{\bar{0}}$.
2. It is obvious that $\dim(\mathfrak{g}_{\bar{1}}) \geq 2$. If $\dim(U) = 1$ then U is a totally isotropic subspace of $\mathfrak{g}_{\bar{1}}$ so there exists a one-dimensional subspace V of $\mathfrak{g}_{\bar{1}}$ such that B is nondegenerate on $U \oplus V$ (see [Bou59]). Let $\mathfrak{g}_{\bar{1}} = (U \oplus V) \overset{\perp}{\oplus} W$ where $W = (U \oplus V)^\perp$ then for all $f \in \phi(W)$ one has:
$$\{f, I\} = \{f, \alpha \otimes pq\} = -\alpha \otimes (\{f, p\}q + p\{f, q\}) = 0.$$
So $W \subset \mathcal{Z}(\mathfrak{g})$. Since B is nondegenerate on W and $\mathfrak{g}$ is reduced then $W = \{0\}$. If $\dim(U) = 2$ then U is nondegenerate or totally isotropic. If U is nondegenerate, let $\mathfrak{g}_{\bar{1}} = U \overset{\perp}{\oplus} W$ where $W = U^\perp$. If U is totally isotropic, let $\mathfrak{g}_{\bar{1}} = (U \oplus V) \overset{\perp}{\oplus} W$ where $W = (U \oplus V)^\perp$ in $\mathfrak{g}_{\bar{1}}$ and B is nondegenerate on $U \oplus V$. In the both cases, similarly as above, one has W a nondegenerate central ideal so $W = \{0\}$. Therefore, $\dim(\mathfrak{g}_{\bar{1}}) = \dim(U) = 2$ if U is nondegenerate and $\dim(\mathfrak{g}_{\bar{1}}) = \dim(U \oplus V) = 4$ if U is totally isotropic.

$\square$

In the sequel, we obtain the classification result.

Proposition 3.3. *Let $\mathfrak{g}$ be a reduced elementary quadratic Lie superalgebra then $\mathfrak{g}$ is i-isomorphic to one of the following Lie superalgebras:*

1. $\mathfrak{g}_i$ $(3 \leq i \leq 6)$ *the reduced singular quadratic Lie algebras of type* S_3 *given in [PU07].*
2. $\mathfrak{g}^s_{4,1} = (\mathbb{C}X_{\overline{0}} \oplus \mathbb{C}Y_{\overline{0}}) \oplus (\mathbb{C}X_{\overline{1}} \oplus \mathbb{C}Z_{\overline{1}})$ *where* $\mathfrak{g}_{\overline{0}} = \operatorname{span}\{X_{\overline{0}}, Y_{\overline{0}}\}$, $\mathfrak{g}_{\overline{1}} = \operatorname{span}\{X_{\overline{1}}, Z_{\overline{1}}\}$, *the bilinear form B is defined by* $B(X_{\overline{0}}, Y_{\overline{0}}) = B(X_{\overline{1}}, Z_{\overline{1}}) = 1$, *the other are zero and the Lie super-bracket is given by* $[Z_{\overline{1}}, Z_{\overline{1}}] = -2X_{\overline{0}}$, $[Y_{\overline{0}}, Z_{\overline{1}}] = -2X_{\overline{1}}$, *the others are trivial.*
3. $\mathfrak{g}^s_{4,2} = (\mathbb{C}X_{\overline{0}} \oplus \mathbb{C}Y_{\overline{0}}) \oplus (\mathbb{C}X_{\overline{1}} \oplus \mathbb{C}Y_{\overline{1}})$ *where* $\mathfrak{g}_{\overline{0}} = \operatorname{span}\{X_{\overline{0}}, Y_{\overline{0}}\}$, $\mathfrak{g}_{\overline{1}} = \operatorname{span}\{X_{\overline{1}}, Y_{\overline{1}}\}$, *the bilinear form B is defined by* $B(X_{\overline{0}}, Y_{\overline{0}}) = B(X_{\overline{1}}, Y_{\overline{1}}) = 1$, *the other are zero and the Lie super-bracket is given by* $[X_{\overline{1}}, Y_{\overline{1}}] = X_{\overline{0}}$, $[Y_{\overline{0}}, X_{\overline{1}}] = X_{\overline{1}}$, $[Y_{\overline{0}}, Y_{\overline{1}}] = -Y_{\overline{1}}$, *the others are trivial.*
4. $\mathfrak{g}^s_{6} = (\mathbb{C}X_{\overline{0}} \oplus \mathbb{C}Y_{\overline{0}}) \oplus (\mathbb{C}X_{\overline{1}} \oplus \mathbb{C}Y_{\overline{1}} \oplus \mathbb{C}Z_{\overline{1}} \oplus \mathbb{C}T_{\overline{1}})$ *where* $\mathfrak{g}_{\overline{0}} = \operatorname{span}\{X_{\overline{0}}, Y_{\overline{0}}\}$, $\mathfrak{g}_{\overline{1}} = \operatorname{span}\{X_{\overline{1}}, Y_{\overline{1}}, Z_{\overline{1}}, T_{\overline{1}}\}$, *the bilinear form B is defined by* $B(X_{\overline{0}}, Y_{\overline{0}}) = 1$, $B(X_{\overline{1}}, Z_{\overline{1}}) = B(Y_{\overline{1}}, T_{\overline{1}}) = 1$, *the other are zero and the Lie super-bracket is given by* $[Z_{\overline{1}}, T_{\overline{1}}] = -X_{\overline{0}}$, $[Y_{\overline{0}}, Z_{\overline{1}}] = -Y_{\overline{1}}$, $[Y_{\overline{0}}, T_{\overline{1}}] = -X_{\overline{1}}$, *the others are trivial.*

Proof.

1. This statement corresponds to the case where I is a decomposable 3-form in $\mathrm{Alt}^3(\mathfrak{g}_{\overline{0}})$. Therefore, the result is obvious.
   Assume that $I = \alpha \otimes pq \in \mathrm{Alt}^1(\mathfrak{g}_{\overline{0}}) \otimes \mathrm{Sym}^2(\mathfrak{g}_{\overline{1}})$. By the previous Lemma, we can write $\mathfrak{g}_{\overline{0}} = \mathbb{C}X_{\overline{0}} \oplus \mathbb{C}Y_{\overline{0}}$ where $X_{\overline{0}} = \phi^{-1}(\alpha)$, $B(X_{\overline{0}}, X_{\overline{0}}) = B(Y_{\overline{0}}, Y_{\overline{0}}) = 0$, $B(X_{\overline{0}}, Y_{\overline{0}}) = 1$. Let $X_{\overline{1}} = \phi^{-1}(p)$, $Y_{\overline{1}} = \phi^{-1}(q)$ and $U = \operatorname{span}\{X_{\overline{1}}, Y_{\overline{1}}\}$.
2. If $\dim(U) = 1$ then $Y_{\overline{1}} = kX_{\overline{1}}$ with some nonzero $k \in \mathbb{C}$. Therefore, $q = kp$ and $I = k\alpha \otimes p^2$. Replacing $X_{\overline{0}}$ by $kX_{\overline{0}}$ and $Y_{\overline{0}}$ by $\frac{1}{k}Y_{\overline{0}}$, we can assume that $k = 1$. Let $Z_{\overline{1}}$ be an element in $\mathfrak{g}_{\overline{1}}$ such that $B(X_{\overline{1}}, Z_{\overline{1}}) = 1$.
   Now, let $X \in \mathfrak{g}_{\overline{0}}, Y, Z \in \mathfrak{g}_{\overline{1}}$. By using (1.7) and (1.8) of [BP89], one has:
   $$B(X, [Y, Z]) = -2\alpha(X)p(Y)p(Z) = -2B(X_{\overline{0}}, X)B(X_{\overline{1}}, Y)B(X_{\overline{1}}, Z).$$
   Since $B|_{\mathfrak{g}_{\overline{0}} \times \mathfrak{g}_{\overline{0}}}$ is nondegenerate and invariant, we have:
   $$[Y, Z] = -2B(X_{\overline{1}}, Y)B(X_{\overline{1}}, Z)X_{\overline{0}} \text{ and } [X, Y] = -2B(X_{\overline{0}}, X)B(X_{\overline{1}}, Y)X_{\overline{1}},$$
   for all $X \in \mathfrak{g}_{\overline{0}}$, Y, $Z \in \mathfrak{g}_{\overline{1}}$, and (2) then follows.
3. If $\dim(U) = 2$ and U is nondegenerate then $B(X_{\overline{1}}, Y_{\overline{1}}) = a \neq 0$. Replacing $X_{\overline{1}}$ by $\frac{1}{a}X_{\overline{1}}$, $X_{\overline{0}}$ by $aX_{\overline{0}}$ and $Y_{\overline{0}}$ by $\frac{1}{a}Y_{\overline{0}}$, we can assume that $a = 1$. Then one has $\mathfrak{g}_{\overline{0}} = \operatorname{span}\{X_{\overline{0}}, Y_{\overline{0}}\}$, $\mathfrak{g}_{\overline{1}} = \operatorname{span}\{X_{\overline{1}}, Y_{\overline{1}}\}$, $B(X_{\overline{0}}, Y_{\overline{0}}) = B(X_{\overline{1}}, Y_{\overline{1}}) = 1$ and $I = \alpha \otimes pq$. Let $X \in \mathfrak{g}_{\overline{0}}$, $Y, Z \in \mathfrak{g}_{\overline{1}}$, we have:
   $$B(X, [Y, Z]) = I(X, Y, Z) = -\alpha(X)(p(Y)q(Z) + p(Z)q(Y)).$$
   Therefore, by a completely similar way as (2) we obtain (3).

4. If $\dim(U) = 2$ and U is totally isotropic: let $V = \mathrm{span}\{Z_{\bar{1}}, T_{\bar{1}}\}$ be a 2-dimensional totally isotropic subspace of $\mathfrak{g}_{\bar{1}}$ such that $\mathfrak{g}_{\bar{1}} = U \oplus V$ and $B(X_{\bar{1}}, Z_{\bar{1}}) = B(Y_{\bar{1}}, T_{\bar{1}}) = 1$. Let $X \in \mathfrak{g}_{\bar{0}}, Y, Z \in \mathfrak{g}_{\bar{1}}$ then:

$$B(X, [Y, Z]) = I(X, Y, Z) = -\alpha(X)(p(Y)q(Z) + p(Z)q(Y)).$$

and therefore (4) follows.

$\square$

4. Quadratic Lie superalgebras with 2-dimensional even part

This Section is devoted to study another particular case of singular quadratic Lie superalgebras: quadratic Lie superalgebras with 2-dimensional even part. As we shall see, they can be seen as a symplectic version of solvable singular quadratic Lie algebras. The first result classifies these algebras using the dup-number.

Proposition 4.1. *Let* $\mathfrak{g}$ *be a non-Abelian quadratic Lie superalgebra with* $\dim(\mathfrak{g}_{\bar{0}}) = 2$*. Then* $\mathfrak{g}$ *is a singular quadratic Lie superalgebra of type* S_1.

Proof. Let I be the associated invariant of $\mathfrak{g}$. By a remark in [PU07], every non-Abelian quadratic Lie algebra must have the dimension more than 2 so $\mathfrak{g}_{\overline{0}}$ is Abelian and as a consequence, $I \in \mathrm{Alt}^1(\mathfrak{g}_{\bar{0}}) \otimes \mathrm{Sym}^2(\mathfrak{g}_{\bar{1}})$. We choose a basis $\{X_{\overline{0}}, Y_{\overline{0}}\}$ of $\mathfrak{g}_{\overline{0}}$ such that $B(X_{\overline{0}}, X_{\overline{0}}) = B(Y_{\overline{0}}, Y_{\overline{0}}) = 0$ and $B(X_{\overline{0}}, Y_{\overline{0}}) = 1$. Let $\alpha = \phi(X_{\overline{0}})$, $\beta = \phi(Y_{\overline{0}})$ and we can assume $I = \alpha \otimes \Omega_1 + \beta \otimes \Omega_2$ where $\Omega_1, \Omega_2 \in \mathrm{Sym}^2(\mathfrak{g}_{\bar{1}})$. Then one has:

$$\{I, I\} = 2(\Omega_1\Omega_2 + \alpha \wedge \beta \otimes \{\Omega_1, \Omega_2\}).$$

Hence, $\{I, I\} = 0$ implies $\Omega_1 = 0$ or $\Omega_2 = 0$ and then $\mathfrak{g}$ is singular of type S_1. $\square$

Proposition 4.2. *Let* $\mathfrak{g}$ *be a singular quadratic Lie superalgebra with Abelian even part. If* $\mathfrak{g}$ *is reduced then* $\dim(\mathfrak{g}_{\overline{0}}) = 2$.

Proof. Let I be the associated invariant of $\mathfrak{g}$ then $I \in \mathrm{Alt}^1(\mathfrak{g}_{\bar{0}}) \otimes \mathrm{Sym}^2(\mathfrak{g}_{\bar{1}})$. Moreover $\mathfrak{g}$ is singular then $I = \alpha \otimes \Omega$ where $\alpha \in \mathfrak{g}_{\bar{0}}^*$, $\Omega \in \mathrm{Sym}^2(\mathfrak{g}_{\bar{1}})$. The proof follows exactly Lemma 3.2. Let $\beta \in \mathfrak{g}_{\bar{0}}^*$ then $\{\beta, \alpha\} = 0$ if and only if $\{\beta, I\} = 0$, equivalently $\phi^{-1}(\beta) \in \mathcal{Z}(\mathfrak{g})$. Therefore, $(\phi^{-1}(\alpha))^\perp \cap \mathfrak{g}_{\bar{0}} \subset \mathcal{Z}(\mathfrak{g})$. It means that $\dim(\mathfrak{g}_{\bar{0}}) = 2$ since $\mathfrak{g}$ is reduced and $\phi^{-1}(\alpha)$ is isotropic in $\mathcal{Z}(\mathfrak{g})$. $\square$

Now, let $\mathfrak{g}$ be a non-Abelian quadratic Lie superalgebra with 2-dimensional even part. By Proposition 4.1, $\mathfrak{g}$ is singular of type S_1. Fix $\alpha \in \mathcal{V}_I$ and choose $\Omega \in \mathrm{Sym}^2(\mathfrak{g})$ such that $I = \alpha \otimes \Omega$. Define $C : \mathfrak{g}_{\bar{1}} \to \mathfrak{g}_{\bar{1}}$ by $B(C(X), Y) = \Omega(X, Y)$, for all $X, Y \in \mathfrak{g}_{\bar{1}}$ and let $X_{\overline{0}} = \phi^{-1}(\alpha)$.

Lemma 4.3. *The following assertions are equivalent:*

1. $\{I,I\} = 0$,
2. $\{\alpha,\alpha\} = 0$,
3. $B(X_{\overline{0}},X_{\overline{0}}) = 0$.

In this case, one has $X_{\overline{0}} \in \mathscr{Z}(\mathfrak{g})$.

Proof. It is easy to see that $\{I,I\} = 0$ if and only if $\{\alpha,\alpha\} \otimes \Omega^2 = 0$. Therefore the assertions are equivalent. Moreover, since $\{\alpha,I\} = 0$ one has $X_{\overline{0}} \in \mathscr{Z}(\mathfrak{g})$. $\square$

Proposition 4.4. *Let $\mathfrak{g}$ be a non-Abelian quadratic Lie superalgebra with* $\dim(\mathfrak{g}_{\overline{0}}) = 2$. *Define $X_{\overline{0}}$ and C as above. Then one has:*

1. *The map C is skew-symmetric, $C = \mathrm{ad}(Y_{\overline{0}})|_{\mathfrak{g}_{\overline{1}}}$ where $Y_{\overline{0}} \in \mathfrak{g}_{\overline{0}}$ is isotropic such that $B(X_{\overline{0}},Y_{\overline{0}}) = 1$, and $[X,Y] = B(C(X),Y)X_{\overline{0}}$, for all $X,Y \in \mathfrak{g}_{\overline{1}}$.*
2. $\mathscr{Z}(\mathfrak{g}) = \ker(C) \oplus \mathbb{C}X_{\overline{0}}$ *and* $[\mathfrak{g},\mathfrak{g}] = \mathrm{Im}(C) \oplus \mathbb{C}X_{\overline{0}}$. *Therefore, $\mathfrak{g}$ is reduced if and only if* $\ker(C) \subset \mathrm{Im}(C)$.
3. *$\mathfrak{g}$ is solvable. Moreover, $\mathfrak{g}$ is nilpotent if and only if C is nilpotent.*

Proof. The assertion (1) is direct by using $\alpha(X) = B(X_{\overline{0}},X)$, $B(X,[Y,Z]) = (\alpha \otimes \Omega)(X,Y,Z)$ and $\Omega(Y,Z) = B(C(Y),Z)$ for all $X \in \mathfrak{g}_{\overline{0}}$, Y, $Z \in \mathfrak{g}_{\overline{1}}$ combining with the properties of B. The statement (2) follows (1). For (3), $\mathfrak{g}$ is solvable since $\mathfrak{g}_{\overline{0}}$ is solvable, or since $[[\mathfrak{g},\mathfrak{g}],[\mathfrak{g},\mathfrak{g}]] \subset \mathbb{C}X_{\overline{0}}$. If $\mathfrak{g}$ is nilpotent then $C = \mathrm{ad}(Y_{\overline{0}})$ is nilpotent obviously. Conversely, if C is nilpotent then it is easy to see that $\mathfrak{g}$ is nilpotent since $(\mathrm{ad}(X))^k(\mathfrak{g}) \subset \mathbb{C}X_{\overline{0}} \oplus \mathrm{Im}(C^k)$ for all $X \in \mathfrak{g}$. $\square$

Remark 4.5. The choice of C is unique up to a nonzero scalar. Indeed, assume that $I = \alpha' \otimes \Omega'$ and C' is the map associated to Ω'. Since $\mathscr{Z}(\mathfrak{g}) \cap \mathfrak{g}_{\overline{0}} = \mathbb{C}X_{\overline{0}}$ and $\phi^{-1}(\alpha') \in \mathscr{Z}(\mathfrak{g})$ one has $\alpha' = \lambda\alpha$ for some nonzero $\lambda \in \mathbb{C}$. Therefore, $\alpha \otimes (\Omega - \lambda\Omega') = 0$. It means that $\Omega = \lambda\Omega'$ and then we get $C = \lambda C'$.

Double extensions are a very useful method initiated by V. Kac to construct quadratic Lie algebras (see [Kac85] and [MR85]). They are generalized to many algebras endowed with a nondegenerate invariant bilinear form, for example quadra tic Lie superalgebras (see [BB99] and [BBB]). In [DPU12], we consider a particular case that is the double extension of a quadratic vector space by a skew-symmetric map. From this we obtain the class of solvable singular quadratic Lie algebras. Here, we use this notion in yet another context, replacing the quadratic vector space by a symplectic vector space.

Definition 4.6.

1. Let $(\mathfrak{q}, B_{\mathfrak{q}})$ be a symplectic vector space equipped with a symplectic bilinear form $B_{\mathfrak{q}}$ and $\overline{C} : \mathfrak{q} \to \mathfrak{q}$ be a skew-symmetric map. $(\mathfrak{t} = \operatorname{span}\{X_{\bar{0}}, Y_{\bar{0}}\}, B_{\mathfrak{t}})$ be a 2-dimensional quadratic vector space with the symmetric bilinear form $B_{\mathfrak{t}}$ defined by $B_{\mathfrak{t}}(X_{\bar{0}}, X_{\bar{0}}) = B_{\mathfrak{t}}(Y_{\bar{0}}, Y_{\bar{0}}) = 0$ and $B_{\mathfrak{t}}(X_{\bar{0}}, Y_{\bar{0}}) = 1$.

   Consider the vector space $\mathfrak{g} = \mathfrak{t} \overset{\perp}{\oplus} \mathfrak{q}$ equipped with the bilinear form $B = B_{\mathfrak{t}} + B_{\mathfrak{q}}$ and define a bracket on $\mathfrak{g}$ by

   $$[\lambda X_{\bar{0}} + \mu Y_{\bar{0}} + X, \lambda' X_{\bar{0}} + \mu' Y_{\bar{0}} + Y] = \mu \overline{C}(Y) - \mu' \overline{C}(X) + B(\overline{C}(X), Y) X_{\bar{0}},$$

   for all $X, Y \in \mathfrak{q}, \lambda, \mu, \lambda', \mu' \in \mathbb{C}$. Then $(\mathfrak{g}, B)$ is a quadratic solvable Lie superalgebra with $\mathfrak{g}_{\bar{0}} = \mathfrak{t}$, $\mathfrak{g}_{\bar{1}} = \mathfrak{q}$. We call $\mathfrak{g}$ the *double extension of* $\mathfrak{q}$ *by* $\overline{C}$ and denote it $(\mathfrak{g}, B, \overline{C})$.
2. Let $\mathfrak{g}_i$ be double extensions of symplectic vector spaces $(\mathfrak{q}_i, B_i)$ by skew-symmetric maps $\overline{C}_i \in \operatorname{End}(\mathfrak{q}_i)$, for $1 \le i \le k$. The *amalgamated product*

   $$\mathfrak{g} = \mathfrak{g}_1 \underset{\mathrm{a}}{\times} \mathfrak{g}_2 \underset{\mathrm{a}}{\times} \dots \underset{\mathrm{a}}{\times} \mathfrak{g}_k$$

   is defined as the double extension of $\mathfrak{q}$ by the skew-symmetric map $\overline{C} \in \operatorname{End}(\mathfrak{q})$ where

   - $\mathfrak{q} = \mathfrak{q}_1 \oplus \dots \oplus \mathfrak{q}_k$ and the bilinear form B such that $B\left(\sum_{i=I}^{k} X_i, \sum_{i=I}^{k} Y_i\right) = \sum_{i=I}^{k} B_i(X_i, Y_i)$, for $X_i, Y_i \in \mathfrak{q}_i$, $1 \le i \le k$.
   - $\overline{C}\left(\sum_{i=I}^{k} X_i\right) = \sum_{i=I}^{k} \overline{C}_i(X_i)$, for $X_i \in \mathfrak{q}_i$, $1 \le i \le k$.

Keeping this notation, the Lemma below is a straight translation of the Lie algebra case, see the proof of Lemma 5.2 in [DPU12].

Lemma 4.7. *We keep the notation above.*

1. *Let $\mathfrak{g}$ be the double extension of $\mathfrak{q}$ by $\overline{C}$. Then*

   $$[X, Y] = B(X_{\bar{0}}, X) C(Y) - B(X_{\bar{0}}, Y) C(X) + B(C(X), Y) X_{\bar{0}}, \ \forall\, X, Y \in \mathfrak{g},$$

   where $C = \operatorname{ad}(Y_{\bar{0}})$. Moreover, $X_{\bar{0}} \in \mathcal{Z}(\mathfrak{g})$ and $C|_{\mathfrak{q}} = \overline{C}$.
2. *Let $\mathfrak{g}'$ be the double extension of $\mathfrak{q}$ by $\overline{C'} = \lambda \overline{C}$, $\lambda \in \mathbb{C}$, $\lambda \neq 0$. Then $\mathfrak{g}$ and $\mathfrak{g}'$ are i-isomorphic.*

Proposition 4.8.

1. *Let $\mathfrak{g}$ be a non-Abelian quadratic Lie superalgebra with 2-dimensional even part. Keep the notations as in Proposition 4.4. Then $\mathfrak{g}$ is the double extension of $\mathfrak{q} = (\mathbb{C}X_{\overline{0}} \oplus \mathbb{C}Y_{\overline{0}})^{\perp} = \mathfrak{g}_{\overline{1}}$ by $\overline{C} = \mathrm{ad}(Y_{\overline{0}})|_{\mathfrak{q}}$.*
2. *Let $\mathfrak{g}$ be the double extension of a symplectic vector space $\mathfrak{q}$ by a map $\overline{C} \neq 0$. Then $\mathfrak{g}$ is a singular solvable quadratic Lie superalgebra with 2-dimensional even part. Moreover:*
   - *(i) $\mathfrak{g}$ is reduced if and only if $\ker(\overline{C}) \subset \mathrm{Im}(\overline{C})$.*
   - *(ii) $\mathfrak{g}$ is nilpotent if and only if $\overline{C}$ is nilpotent.*
3. *Let $(\mathfrak{g}, B)$ be a quadratic Lie superalgebra. Let $\mathfrak{g}'$ be the double extension of a symplectic vector space $(\mathfrak{q}', B')$ by a map $\overline{C'}$. Let A be an i-isomorphism of $\mathfrak{g}'$ onto $\mathfrak{g}$ and write $\mathfrak{q} = A(\mathfrak{q}')$. Then $\mathfrak{g}$ is the double extension of $(\mathfrak{q}, B|_{\mathfrak{q}\times\mathfrak{q}})$ by the map $\overline{C} = \overline{A}\,\overline{C'}\,\overline{A}^{-1}$ where $\overline{A} = A|_{\mathfrak{q}'}$.*

Proof. The assertions (1) and (2) follow Proposition 4.4 and Lemma 4.7. For (3), since A is i-isomorphic then $\mathfrak{g}$ has also 2-dimensional even part. Write $\mathfrak{g}' = (\mathbb{C}X'_{\overline{0}} \oplus \mathbb{C}Y'_{\overline{0}}) \overset{\perp}{\oplus} \mathfrak{q}'$. Let $X_{\overline{0}} = A(X'_{\overline{0}})$ and $Y_{\overline{0}} = A(Y'_{\overline{0}})$. Then $\mathfrak{g} = (\mathbb{C}X_{\overline{0}} \oplus \mathbb{C}Y_{\overline{0}}) \overset{\perp}{\oplus} \mathfrak{q}$ and one has:

$$[Y_{\overline{0}}, X] = (A\overline{C'}A^{-1})(X),\ \forall\, X \in \mathfrak{q},\ \text{and}$$

$$[X, Y] = B((A\overline{C'}A^{-1})(X), Y)X_{\overline{0}},\ \forall\, X, Y \in \mathfrak{q}.$$

This proves the result. □

Example 4.9. From the point of view of double extensions, for reduced elementary quadratic Lie superalgebras with 2-dimensional even part in Section 3 one has

1. $\mathfrak{g}^s_{4,1}$ and $\mathfrak{g}^s_{4,2}$ are double extensions of the 2-dimensional symplectic vector space $\mathfrak{q} = \mathbb{C}^2$ by maps having matrices $\overline{C}_1 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ and $\overline{C}_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, respectively, in a Darboux basis $\{E_1, E_2\}$ of $\mathfrak{q}$ where $B(E_1, E_2) = 1$.
2. $\mathfrak{g}^s_6$ is the double extension of the 4-dimensional symplectic vector space $\mathfrak{q} = \mathbb{C}^4$ by the map having matrix:

$$\overline{C} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{pmatrix}$$

   in a Darboux basis $\{E_1, E_2, E_3, E_4\}$ of $\mathfrak{q}$ where $B(E_1, E_3) = B(E_2, E_4) = 1$, the other are zero.

Let $(\mathfrak{q}, B)$ be a symplectic vector space. Denote by $\mathrm{Sp}(\mathfrak{q})$ the isometry group of B and by $\mathfrak{sp}(\mathfrak{q})$ its Lie algebra, i.e. the Lie algebra of skew-symmetric maps with respects to B. The *adjoint action* is the action of $\mathrm{Sp}(\mathfrak{q})$ on $\mathfrak{sp}(\mathfrak{q})$ by conjugation (see Appendix). Also, we denote by $\mathbb{P}^1(\mathfrak{sp}(2n))$ the projective space of $\mathfrak{sp}(2n)$ with the action induced by $\mathrm{Sp}(2n)$-adjoint action on $\mathfrak{sp}(2n)$.

Proposition 4.10. *Let $(\mathfrak{q}, B)$ be a symplectic vector space. Let $\mathfrak{g} = (\mathbb{C}X_{\bar{0}} \oplus \mathbb{C}Y_{\bar{0}}) \overset{\perp}{\oplus} \mathfrak{q}$ and $\mathfrak{g}' = (\mathbb{C}X'_{\bar{0}} \oplus \mathbb{C}Y'_{\bar{0}}) \overset{\perp}{\oplus} \mathfrak{q}$ be double extensions of $\mathfrak{q}$, by skew-symmetric maps $\overline{C}$ and $\overline{C'}$ respectively. Then:*

1. *there is a Lie superalgebra isomorphism from $\mathfrak{g}$ onto $\mathfrak{g}'$ if and only if there are an invertible $P \in \mathrm{End}(\mathfrak{q})$ and a nonzero $\lambda \in \mathbb{C}$ such that $\overline{C'} = \lambda\, P\overline{C}P^{-1}$ and $P^*P\overline{C} = \overline{C}$ where P^* is the adjoint map of P with respect to B.*
2. *there exists an i-isomorphism between $\mathfrak{g}$ and $\mathfrak{g}'$ if and only if $\overline{C'}$ is in the $\mathrm{Sp}(\mathfrak{q})$-adjoint orbit through $\lambda\overline{C}$ for some nonzero $\lambda \in \mathbb{C}$.*

Proof. The assertions are obvious if $\overline{C} = 0$. We assume $\overline{C} \neq 0$.

1. Let $A : \mathfrak{g} \to \mathfrak{g}'$ be a Lie superalgebra isomorphism then $A(\mathbb{C}X_{\bar{0}} \oplus \mathbb{C}Y_{\bar{0}}) = \mathbb{C}X'_{\bar{0}} \oplus \mathbb{C}Y'_{\bar{0}}$ and $A(\mathfrak{q}) = \mathfrak{q}$. It is obvious that $\overline{C'} \neq 0$. It is easy to see that $\mathbb{C}X_{\bar{0}} = \mathscr{Z}(\mathfrak{g}) \cap \mathfrak{g}_{\bar{0}}$ and $\mathbb{C}X'_{\bar{0}} = \mathscr{Z}(\mathfrak{g}') \cap \mathfrak{g}'_{\bar{0}}$ then one has $A(\mathbb{C}X_{\bar{0}}) = \mathbb{C}X'_{\bar{0}}$. It means $A(X_{\bar{0}}) = \mu X'_{\bar{0}}$ for a nonzero $\mu \in \mathbb{C}$. Let $A|_{\mathfrak{q}} = Q$ and assume $A(Y_{\bar{0}}) = \beta Y'_{\bar{0}} + \gamma X'_{\bar{0}}$. For all $X, Y \in \mathfrak{q}$, we have $A([X,Y]) = \mu B(\overline{C}(X), Y) X'_{\bar{0}}$. Also, $A([X,Y]) = [Q(X), Q(Y)]' = B(\overline{C'}Q(X), Q(Y)) X'_{\bar{0}}$. Then $Q^*\overline{C'}Q = \mu\overline{C}$.
   Moreover, $A([Y_{\bar{0}}, X]) = Q(\overline{C}(X)) = [\beta Y'_{\bar{0}} + \gamma X'_{\bar{0}}, Q(X)]' = \beta\overline{C'}Q(X)$, for all $X \in \mathfrak{q}$. We conclude that $Q\ \overline{C}\ Q^{-1} = \beta\overline{C'}$ and since $Q^*\overline{C'}Q = \mu\overline{C}$, then $Q^*Q\overline{C} = \beta\mu\overline{C}$.
   Set $P = \dfrac{1}{(\mu\beta)^{\frac{1}{2}}} Q$ and $\lambda = \dfrac{1}{\beta}$. Then $\overline{C'} = \lambda P\overline{C}P^{-1}$ and $P^*P\overline{C} = \overline{C}$.

   Conversely, assume $\mathfrak{g} = (\mathbb{C}X_{\bar{0}} \oplus \mathbb{C}Y_{\bar{0}}) \overset{\perp}{\oplus} \mathfrak{q}$ and $\mathfrak{g}' = (\mathbb{C}X'_{\bar{0}} \oplus \mathbb{C}Y'_{\bar{0}}) \overset{\perp}{\oplus} \mathfrak{q}$ are double extensions of $\mathfrak{q}$, respectively by maps $\overline{C}$ and $\overline{C'}$ satisfying $\overline{C'} = \lambda P\overline{C}P^{-1}$ and $P^*P\overline{C} = \overline{C}$ with invertible $P \in \mathrm{End}(\mathfrak{q})$ and nonzero $\lambda \in \mathbb{C}$. Define $A : \mathfrak{g} \to \mathfrak{g}'$ by $A(X_{\bar{0}}) = \lambda X'_{\bar{0}}$, $A(Y_{\bar{0}}) = \dfrac{1}{\lambda} Y'_{\bar{0}}$ and $A(X) = P(X)$, for all $X \in \mathfrak{q}$, then it is easy to check that A is a Lie superalgebra isomorphism.
2. If $\mathfrak{g}$ and $\mathfrak{g}'$ are i-isomorphic, then the isomorphism A in the proof of (1) is an isometry. Hence $P \in \mathrm{Sp}(\mathfrak{q})$ and $\overline{C'} = \lambda P\overline{C}P^{-1}$ gives the result.
   Conversely, define A as above (the sufficiency of (1)). Then A is an isometry and it is easy to check that A is an i-isomorphism.

$\square$

Corollary 4.11. *Let $(\mathfrak{g}, B, \overline{C})$ and $(\mathfrak{g}', B', \overline{C'})$ be double extensions of $(\mathfrak{q}, \overline{B})$ and $(\mathfrak{q}', \overline{B'})$ respectively where $\overline{B} = B|_{\mathfrak{q}\times\mathfrak{q}}$ and $\overline{B'} = B'|_{\mathfrak{q}'\times\mathfrak{q}'}$. Write $\mathfrak{g} = (\mathbb{C}X_{\bar{0}} \oplus \mathbb{C}Y_{\bar{0}}) \overset{\perp}{\oplus} \mathfrak{q}$ and $\mathfrak{g}' = (\mathbb{C}X'_{\bar{0}} \oplus \mathbb{C}Y'_{\bar{0}}) \overset{\perp}{\oplus} \mathfrak{q}'$. Then:*

1. *there exists an i-isomorphism between $\mathfrak{g}$ and $\mathfrak{g}'$ if and only if there exists an isometry $\overline{A} : \mathfrak{q} \to \mathfrak{q}'$ such that $\overline{C'} = \lambda\, \overline{A}\, \overline{C}\, \overline{A}^{-1}$, for some nonzero $\lambda \in \mathbb{C}$.*
2. *there exists a Lie superalgebra isomorphism between $\mathfrak{g}$ and $\mathfrak{g}'$ if and only if there exist invertible maps $\overline{Q} : \mathfrak{q} \to \mathfrak{q}'$ and $\overline{P} \in \mathrm{End}(\mathfrak{q})$ such that*

   (i) $\overline{C'} = \lambda\, \overline{Q}\, \overline{C}\, \overline{Q}^{-1}$ *for some nonzero $\lambda \in \mathbb{C}$,*

   (ii) $\overline{P}^*\, \overline{P}\, \overline{C} = \overline{C}$ *and $\overline{Q}\, \overline{P}^{-1}$ is an isometry from $\mathfrak{q}$ onto $\mathfrak{q}'$.*

Proof. 1. Assume $\dim(\mathfrak{g}) = \dim(\mathfrak{g}')$ and define $F : \mathfrak{g}' \to \mathfrak{g}$ by $F(X'_{\bar{0}}) = X_{\bar{0}}$, $F(Y'_{\bar{0}}) = Y_{\bar{0}}$ and $\overline{F} = F|_{\mathfrak{q}'}$ an isometry from $\mathfrak{q}'$ onto $\mathfrak{q}$. Setting a new Lie bracket on $\mathfrak{g}$ by $[X,Y]'' = F\left([F^{-1}(X), F^{-1}(Y)]'\right)$, for all $X, Y \in \mathfrak{g}$ defines a new Lie superalgebra $\mathfrak{g}''$. So F is an i-isomorphism from $\mathfrak{g}'$ onto $\mathfrak{g}''$.

Moreover $\mathfrak{g}'' = (\mathbb{C}X_1 \oplus \mathbb{C}Y_1) \overset{\perp}{\oplus} \mathfrak{q}$ is the double extension of $\mathfrak{q}$ by $\overline{C''}$ with $\overline{C''} = \overline{F}\, \overline{C'}\, \overline{F}^{-1}$ by Proposition 4.8 (3). Then $\mathfrak{g}$ and $\mathfrak{g}'$ are i-isomorphic if and only if $\mathfrak{g}$ and $\mathfrak{g}''$ are i-isomorphism. Applying Proposition 4.10, this is the case if and only if there exists $\overline{A} \in \mathrm{Sp}(\mathfrak{q})$ such that $\overline{C''} = \lambda\, \overline{A}\, \overline{C}\, \overline{A}^{-1}$ for some nonzero complex λ. That implies $\overline{C'} = \lambda\, (\overline{F}^{-1}\, \overline{A})\, \overline{C}\, (\overline{F}^{-1}\overline{A})^{-1}$.

2. Keep the notation as in (1). We have that $\mathfrak{g}$ and $\mathfrak{g}'$ are isomorphic if and only if $\mathfrak{g}$ and $\mathfrak{g}''$ are isomorphic. By Proposition 4.10, $\mathfrak{g}$ and $\mathfrak{g}''$ are isomorphic if and only if there exists an invertible map $\overline{P} \in \mathrm{End}(\mathfrak{q})$ and a nonzero $\lambda \in \mathbb{C}$ such that $\overline{C''} = \lambda\, \overline{P}\, \overline{C}\, \overline{P}^{-1}$ and $\overline{P}^*\, \overline{P}\, \overline{C} = \overline{C}$. We conclude that $\overline{C'} = \lambda\, \overline{Q}\, \overline{C}\, \overline{Q}^{-1}$ with $\overline{Q} = \overline{F}^{-1}\, \overline{P}$. Finally, $\overline{F}^{-1} = \overline{Q}\, \overline{P}^{-1}$ is an isometry from $\mathfrak{q}$ to $\mathfrak{q}'$.
   On the other hand, if $\overline{C'} = \lambda\, \overline{Q}\, \overline{C}\, \overline{Q}^{-1}$ and $\overline{P}^*\, \overline{P}\, \overline{C} = \overline{C}$ with $\overline{P} = \overline{F}\, \overline{Q}$ for some isometry $\overline{F} : \mathfrak{q}' \to \mathfrak{q}$, then construct $\mathfrak{g}''$ as in (1). We deduce $\overline{C''} = \lambda\, \overline{P}\, \overline{C}\, \overline{P}^{-1}$ and $\overline{P}^*\, \overline{P}\, \overline{C} = \overline{C}$. So, by Proposition 4.10, $\mathfrak{g}$ and $\mathfrak{g}''$ are isomorphic and therefore, $\mathfrak{g}$ and $\mathfrak{g}'$ are isomorphic.

□

It results that quadratic Lie superalgebra structures on the quadratic $\mathbb{Z}_2$-graded vector space $\mathbb{C}^2 \underset{\mathbb{Z}_2}{\oplus} \mathbb{C}^{2n}$ can be classified up to i-isomorphism in terms of $\mathrm{Sp}(2n)$-orbits in $\mathbb{P}^1(\mathfrak{sp}(2n))$. Next, we give an explicit classification of $\mathrm{Sp}(2n)$-adjoint orbits of $\mathfrak{sp}(2n)$ and recall the classification of $\mathrm{O}(n)$-adjoint orbits of $\mathfrak{o}(n)$ in [DPU12] (see also the Appendix).

Remark 4.12. Given $n \in \mathbb{N}^*$, a *partition* $[d]$ of n is a tuple $[d_1,...,d_k]$ of positive integers satisfying $d_1 \geq ... \geq d_k$ and $d_1 + ... + d_k = n$. Denote by $\mathscr{P}(n)$ the set of partitions of n and with $\varepsilon \in \{-1,1\}$ define the sets

$$\mathscr{P}_\varepsilon(n) = \{[d_1,...,d_k] \in \mathscr{P}(n) |\ \sharp\{j \mid\ d_j = i\} \text{ is even for all i such that } (-1)^i = \varepsilon\}.$$

It is well-known that nilpotent $\mathrm{O}(n)$-adjoint orbits of $\mathfrak{o}(n)$ and $\mathrm{Sp}(2n)$-adjoint orbits of $\mathfrak{sp}(2n)$ are respectively in one-to-one correspondence with $\mathscr{P}_1(n)$ and $\mathscr{P}_{-1}(2n)$.

For the semisimple case, denote by $\Lambda_n = \{(\lambda_1,\ldots,\lambda_n) \mid\ \lambda_1,\ldots,\lambda_n \in \mathbb{C},\ \lambda_i \neq 0 \ \text{ for some } \ i\}$ and by G_n the group is of all coordinate permutations and sign changes of $(\lambda_1,\ldots,\lambda_n)$. Then there are a bijection between nonzero semisimple $\mathrm{O}(n)$-adjoint orbits of $\mathfrak{o}(n)$ and $\Lambda_{[n/2]}/G_{[n/2]}$ and a bijection between nonzero semisimple $\mathrm{Sp}(2n)$-adjoint orbits of $\mathfrak{sp}(2n)$ and Λ_n/G_n.

For the invertible orbits, i.e. the adjoint orbits of invertible elements, the classification results for $\mathfrak{o}(2n)$ and $\mathfrak{sp}(2n)$ are analogous. We consider

$$\mathscr{D} = \bigcup_{r\in\mathbb{N}^*} \{(d_1,\ldots,d_r) \in \mathbb{N}^r \mid\ d_1 \geq d_2 \geq \cdots \geq d_r \geq 1\}$$

and $\Phi : \mathscr{D} \to \mathbb{N}$ the map defined by $\Phi(d_1,\ldots,d_r) = \sum_{i=1}^r d_i$. Let $\mathscr{J}_n$ be the set of all triples (Λ, b, d) such that:

1. Λ is a subset of $\mathbb{C} \setminus \{0\}$ with $\sharp\Lambda \leq 2n$ and $\lambda \in \Lambda$ if and only if $-\lambda \in \Lambda$.
2. $b : \Lambda \to \mathbb{N}^*$ satisfies $b(\lambda) = b(-\lambda)$, for all $\lambda \in \Lambda$ and $\sum\limits_{\lambda\in\Lambda} b(\lambda) = 2n$.
3. $d : \Lambda \to \mathscr{D}$ satisfies $d(\lambda) = d(-\lambda)$, for all $\lambda \in \Lambda$ and $\Phi \circ d = b$.

Let $\mathscr{I}(2n)$ be the set of invertible elements in $\mathfrak{sp}(2n)$ and $\widetilde{\mathscr{I}}(2n)$ be the set of $\mathrm{Sp}(2n)$-adjoint orbits of elements in $\mathscr{I}(2n)$. Then there is a bijection between $\widetilde{\mathscr{I}}(2n)$ and $\mathscr{J}_n$.

By the foregoing and using the Fitting decomposition, if we denote by $\mathscr{D}(n)$ the set of all pairs $([d],T)$ such that $[d] \in \mathscr{P}_\varepsilon(q)$ (the index $\varepsilon = -1$ for $\mathfrak{sp}(2n)$ and $\varepsilon = 1$ for $\mathfrak{o}(n)$) and $T \in \mathscr{J}_\ell$ satisfying $q + 2\ell = n$ and by $\mathscr{O}(\mathfrak{sp}(2n))$ (resp. $\mathscr{O}(\mathfrak{o}(n))$) the set of $\mathrm{Sp}(2n)$-adjoint orbits of $\mathfrak{sp}(2n)$ (resp. $\mathrm{O}(n)$-adjoint orbits of $\mathfrak{o}(n)$) then there is a bijection between $\mathscr{O}(\mathfrak{sp}(2n))$ and $\mathscr{D}(2n)$ (resp. $\mathscr{O}(\mathfrak{o}(n))$ and $\mathscr{D}(n)$).

Returning to the classification of quadratic Lie superalgebra structures on the quadratic $\mathbb{Z}_2$-graded vector space $\mathbb{C}^2 \underset{\mathbb{Z}_2}{\oplus} \mathbb{C}^{2n}$, we shall need the following lemma:

Lemma 4.13. *Let V be a quadratic $\mathbb{Z}_2$-graded vector space such that its even part is 2-dimensional. We write $V = (\mathbb{C}X'_{\bar{0}} \oplus \mathbb{C}Y'_{\bar{0}}) \overset{\perp}{\oplus} \mathfrak{q}'$ with $X'_{\bar{0}}$, $Y'_{\bar{0}}$ isotropic elements in $V_{\bar{0}}$ and $B(X'_{\bar{0}}, Y'_{\bar{0}}) = 1$. Let $\mathfrak{g}$ be a quadratic Lie superalgebra with* $\dim(\mathfrak{g}_{\bar{0}}) = \dim(V_{\bar{0}})$ *and* $\dim(\mathfrak{g}) = \dim(V)$. *Then, there exists a skew-symmetric map $\overline{C'} : \mathfrak{q}' \to \mathfrak{q}'$ such that V is considered as the double extension of $\mathfrak{q}'$ by $\overline{C'}$ that is i-isomorphic to $\mathfrak{g}$.*

Proof. By Proposition 4.8, $\mathfrak{g}$ is a double extension. Let us write $\mathfrak{g} = (\mathbb{C}X_{\bar{0}} \oplus \mathbb{C}Y_{\bar{0}}) \overset{\perp}{\oplus} \mathfrak{q}$ and $\overline{C} = \mathrm{ad}(Y_{\bar{0}})|_{\mathfrak{q}}$. Define $A : \mathfrak{g} \to V$ by $A(X_{\bar{0}}) = X'_{\bar{0}}$, $A(Y_{\bar{0}}) = Y'_{\bar{0}}$ and $\overline{A} = A|_{\mathfrak{q}}$ any isometry from $\mathfrak{q} \to \mathfrak{q}'$. It is clear that A is an isometry from $\mathfrak{g}$ to V. Now, define the Lie super-bracket on V by $[X,Y] = A\left([A^{-1}(X), A^{-1}(Y)]\right)$, for all $X,Y \in V$. Then V is a quadratic Lie superalgebra, that is i-isomorphic to $\mathfrak{g}$. Moreover, V is obviously the double extension of $\mathfrak{q}'$ by $\overline{C'} = \overline{A}\,\overline{C}\,\overline{A}^{-1}$. $\square$

Proposition 4.8, Proposition 4.10, Corollary 4.11 and Lemma 4.13 are enough to apply the classification method in [DPU12] for the set $\mathcal{S}(2+2n)$ of quadratic Lie superalgebra structures on the quadratic $\mathbb{Z}_2$-graded vector space $\mathbb{C}^2 \underset{\mathbb{Z}_2}{\oplus} \mathbb{C}^{2n}$ by only replacing the isometry group $\mathrm{O}(m)$ by $\mathrm{Sp}(2n)$ and $\mathfrak{o}(m)$ by $\mathfrak{sp}(2n)$ to obtain completely similar results. One has the first characterization of the set $\mathcal{S}(2+2n)$:

Proposition 4.14. *Let $\mathfrak{g}$ and $\mathfrak{g}' \in \mathcal{S}(2+2n)$. Then $\mathfrak{g}$ and $\mathfrak{g}'$ are i-isomorphic if and only if they are isomorphic.*

Using the notion of double extension, we call the Lie superalgebra $\mathfrak{g} \in \mathcal{S}(2+2n)$ *diagonalizable* (resp. *invertible*) if it is a double extension by a diagonalizable (resp. invertible) map. Denote the subsets of nilpotent elements, diagonalizable elements and invertible elements in $\mathcal{S}(2+2n)$, respectively by $\mathcal{N}(2+2n)$, $\mathcal{D}(2+2n)$ and by $\mathcal{S}_{\mathrm{inv}}(2+2n)$. Denote by $\widehat{\mathcal{N}}(2+2n)$, $\widehat{\mathcal{D}}(2+2n)$, $\widehat{\mathcal{S}_{\mathrm{inv}}}(2+2n)$ respectively the sets of isomorphism classes in $\mathcal{N}(2+2n)$, $\mathcal{D}(2+2n)$, $\mathcal{S}_{\mathrm{inv}}(2+2n)$ and $\widehat{\mathcal{D}}_{\mathrm{red}}(2+2n)$ the subset of $\widehat{\mathcal{D}}(2+2n)$ including reduced ones. We have the classification result of these sets as follows:

Proposition 4.15.

1. *There is a bijection between $\widehat{\mathcal{N}}(2+2n)$ and the set of nilpotent $\mathrm{Sp}(2n)$-adjoint orbits of $\mathfrak{sp}(2n)$ that induces a bijection between $\widehat{\mathcal{N}}(2+2n)$ and the set of partitions $\mathscr{P}_{-1}(2n)$.*
2. *There is a bijection between $\widehat{\mathcal{D}}(2+2n)$ and the set of semisimple $\mathrm{Sp}(2n)$-orbits of $\mathbb{P}^1(\mathfrak{sp}(2n))$ that induces a bijection between $\widehat{\mathcal{D}}(2+2n)$ and Λ_n/H_n with H_n the group obtained from G_n by adding maps $(\lambda_1,\dots,\lambda_n) \mapsto \lambda(\lambda_1,\dots,\lambda_n)$, $\forall\lambda \in \mathbb{C}$, $\lambda \neq 0$. In the reduced case, $\widehat{\mathcal{D}}_{\mathrm{red}}(2+2n)$ is bijective to Λ_n^+/H_n with $\Lambda_n^+ = \{(\lambda_1,\dots,\lambda_n) \mid \lambda_1,\dots,\lambda_n \in \mathbb{C},\ \lambda_i \neq 0,\ \forall i\}$.*
3. *There is a bijection between $\widehat{\mathcal{S}_{\mathrm{inv}}}(2+2n)$ and the set of invertible $\mathrm{Sp}(2n)$-orbits of $\mathbb{P}^1(\mathfrak{sp}(2n))$ that induces a bijection between $\widehat{\mathcal{S}_{\mathrm{inv}}}(2+2n)$ and $\mathscr{J}_n/\mathbb{C}^*$.*
4. *There is a bijection between $\widehat{\mathcal{S}}(2+2n)$ and the set of $\mathrm{Sp}(2n)$-orbits of $\mathbb{P}^1(\mathfrak{sp}(2n))$ that induces a bijection between $\widehat{\mathcal{S}}(2+2n)$ and $\mathscr{D}(2n)/\mathbb{C}^*$.*

Proof. Let $\mathfrak{g} \in \mathcal{S}(2+2n)$. If $\mathfrak{g}$ is nilpotent or diagonalizable then the classification of $\mathfrak{g}$ follows the classification of nilpotent and semisimple $\mathrm{Sp}(2n)$-adjoint orbits of $\mathfrak{sp}(2n)$ and therefore we get the statements (1) and (2). For (3), if $\mathfrak{g}$ is invertible then it is a double extension of an invertible $C \in \mathfrak{sp}(2n)$. In this case, the classification of $\widehat{\mathcal{S}_{\mathrm{inv}}}(2+2n)$ can be deduced from the classification of the set of orbits $\widetilde{\mathscr{I}}(2n)$ by $\mathscr{J}_n$ as follows: add an action of the multipplicative group $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$ on $\mathscr{J}_n$ given by $\mu \cdot (\Lambda, b, d) = (\mu\Lambda, b', d')$, $\forall\ (\Lambda, b, d) \in \mathscr{J}_n, \lambda \in \Lambda, \mu \in \mathbb{C}^*$ where $b'(\mu\lambda) = b(\lambda), d'(\mu\lambda) = d(\lambda)$, $\forall\ \lambda \in \Lambda$. Hence, there is a bijection $\widehat{i} : \mathbb{P}^1(\widetilde{\mathscr{I}}(2n)) \to \mathscr{J}_n/\mathbb{C}^*$ given by $\widehat{i}([C]) = [i(C)]$, if $[C]$ is the class of $C \in \mathscr{I}(2n)$ and $[(\Lambda, b, d)]$ is the class of $(\Lambda, b, d) \in \mathscr{J}_n$.

Any $\mathfrak{g} \in \mathcal{S}(2+2n)$ can be decomposed as an amalgamated product of its Fitting components, $\mathfrak{g} = \mathfrak{g}_N \underset{\mathrm{a}}{\times} \mathfrak{g}_I$ where $\mathfrak{g}_N \in \mathcal{N}(2+2k)$ and $\mathfrak{g}_I \in \mathcal{S}_{\mathrm{inv}}(2+2\ell)$, $k, \ell \in \mathbb{N}$ and $2k+2\ell = 2n$. We add an action of the multipplicative group $\mathbb{C}^*$ on the set $\mathscr{D}(2n)$ by $\mu.([d], T) = ([d], \mu.T)$ for all $\mu \in \mathbb{C}^*$, $([d], T) \in \mathscr{D}(2n)$ where $\mu.T$ is defined by the action of the multipplicative group $\mathbb{C}^*$ on $\mathscr{J}_n$. So, the classification of $\mathfrak{g}$ is directly deduced from the classifications in the nilpotent and invertible cases. $\square$

Next, we will describe the sets $\mathcal{N}(2+2n)$, $\mathcal{D}_{\mathrm{red}}(2+2n)$ the subset of $\mathcal{D}(2+2n)$ including reduced ones, and $\mathcal{S}_{\mathrm{inv}}(2+2n)$ in term of amalgamated product in Definition 4.6. Remark that except for the nilpotent case, the amalgamated product may have a bad behavior with respect to isomorphisms.

Definition 4.16. Given $p \in \mathbb{N}^*$. We denote the *Jordan block of size p* by $J_1 = (0)$ and for $p \geq 2$,

$$J_p := \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \dots & \ddots & \vdots \\ 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & 0 & \dots & 0 \end{pmatrix}.$$

We define two types of double extension as follows:

- for $p \geq 2$, we consider the symplectic vector space $\mathfrak{q} = \mathbb{C}^{2p}$ equipped with its canonical bilinear form $\overline{B}$ and the map $\overline{C}^J_{2p}$ having matrix $\begin{pmatrix} J_p & 0 \\ 0 & -{}^tJ_p \end{pmatrix}$ in a Darboux basis. Then $\overline{C}^J_{2p} \in \mathfrak{sp}(2p)$ and we denote by $\mathfrak{j}_{2p}$ the double extension of $\mathfrak{q}$ by $\overline{C}^J_{2p}$. So $\mathfrak{j}_{2p} \in \mathcal{N}(2+2p)$.
- for $p \geq 1$, we consider the symplectic vector space $\mathfrak{q} = \mathbb{C}^{2p}$ equipped with its canonical bilinear form $\overline{B}$ and the map $\overline{C}^J_{p+p}$ with matrix $\begin{pmatrix} J_p & M \\ 0 & -{}^tJ_p \end{pmatrix}$ in a

Darboux basis where $M = (m_{ij})$ denotes the $p \times p$-matrix with $m_{p,p} = 1$ and $m_{ij} = 0$ otherwise. Then $\overline{C}^J_{p+p} \in \mathfrak{sp}(2p)$ and we denote by $\mathfrak{j}_{p+p}$ the double extension of $\mathfrak{q}$ by $\overline{C}^J_{p+p}$. So $\mathfrak{j}_{p+p} \in \mathcal{N}(2+2p)$.

We call $\mathfrak{j}_{2p}$ or $\mathfrak{j}_{p+p}$ *nilpotent Jordan-type* Lie superalgebras.

For $n \in \mathbb{N}^*$, each $[d] \in \mathscr{P}_{-1}(2n)$ can be reordered as $[d] = (p_1, p_1, p_2, p_2, \dots, p_k, p_k, 2q_1, \dots 2q_\ell)$ with all p_i odd, $p_1 \geq p_2 \geq \dots \geq p_k$ and $q_1 \geq q_2 \geq \dots \geq q_\ell$. We associate the partition $[d]$ with the map $\overline{C}_{[d]} \in \mathfrak{sp}(2n)$ having matrix

$$\mathrm{diag}_{k+\ell}(\overline{C}^J_{2p_1}, \overline{C}^J_{2p_2}, \dots, \overline{C}^J_{2p_k}, \overline{C}^J_{q_1+q_1}, \dots, \overline{C}^J_{q_\ell+q_\ell})$$

in a Darboux basis of $\mathbb{C}^{2n}$ and denote by $\mathfrak{g}_{[d]}$ the double extension of $\mathbb{C}^{2n}$ by $\overline{C}_{[d]}$. Then $\mathfrak{g}_{[d]} \in \mathcal{N}(2+2n)$ and $\mathfrak{g}_{[d]}$ is an amalgamated product of nilpotent Jordan-type Lie superalgebras. More precisely,

$$\mathfrak{g}_{[d]} = \mathfrak{j}_{2p_1} \underset{\mathrm{a}}{\times} \mathfrak{j}_{2p_2} \underset{\mathrm{a}}{\times} \dots \underset{\mathrm{a}}{\times} \mathfrak{j}_{2p_k} \underset{\mathrm{a}}{\times} \mathfrak{j}_{q_1+q_1} \underset{\mathrm{a}}{\times} \dots \underset{\mathrm{a}}{\times} \mathfrak{j}_{q_\ell+q_\ell}.$$

Proposition 4.17. *Each $\mathfrak{g} \in \mathcal{N}(2+2n)$ is i-isomorphic to a unique amalgamated product $\mathfrak{g}_{[d]}$, $[d] \in \mathscr{P}_{-1}(2n)$, of nilpotent Jordan-type Lie superalgebras.*

For the reduced diagonalizable case, let $\mathfrak{g}^s_4(\lambda)$ be the double extension of $\mathfrak{q} = \mathbb{C}^2$ by $\overline{C} = \begin{pmatrix} \lambda & 0 \\ 0 & -\lambda \end{pmatrix}$, $\lambda \neq 0$. By Lemma 4.7, $\mathfrak{g}^s_4(\lambda)$ is i-isomorphic to $\mathfrak{g}^s_4(1) = \mathfrak{g}^s_{4,2}$.

Proposition 4.18. *Let $\mathfrak{g} \in \mathcal{D}_{\mathrm{red}}(2+2n)$ then $\mathfrak{g}$ is an amalgamated product of quadratic Lie superalgebras all i-isomorphic to $\mathfrak{g}^s_{4,2}$.*

Finally, for the invertible case, we define the matrix $J_p(\lambda) = \mathrm{diag}_p(\lambda, \dots, \lambda) + J_p$, $p \geq 1$, $\lambda \in \mathbb{C}$ and set

$$\overline{C}^J_{2p}(\lambda) = \begin{pmatrix} J_p(\lambda) & 0 \\ 0 & -{}^tJ_p(\lambda) \end{pmatrix}$$

in a Darboux basis of $\mathbb{C}^{2p}$ then $\overline{C}^J_{2p}(\lambda) \in \mathfrak{sp}(2p)$. Let $\mathfrak{j}_{2p}(\lambda)$ be the double extension of $\mathbb{C}^{2p}$ by $\overline{C}^J_{2p}(\lambda)$ then it is called a *Jordan-type* quadratic Lie superalgebra.

When $\lambda = 0$ and $p \geq 2$, we recover the nilpotent Jordan-type Lie superalgebras $\mathfrak{j}_{2p}$. If $\lambda \neq 0$, $\mathfrak{j}_{2p}(\lambda)$ becomes an invertible singular quadratic Lie superalgebra and

$$\mathfrak{j}_{2p}(-\lambda) \simeq \mathfrak{j}_{2p}(\lambda).$$

Proposition 4.19. *If $\mathfrak{g} \in \mathcal{S}_{\mathrm{inv}}(2+2n)$ then $\mathfrak{g}$ is an amalgamated product of quadratic Lie superalgebras all i-isomorphic to Jordan-type quadratic Lie superalgebras $\mathfrak{j}_{2p}(\lambda)$, with $\lambda \neq 0$.*

Next, we recall the notion of quadratic dimension of quadratic Lie superalgebras (see [Ben03] for more details). Let $(\mathfrak{g}, B)$ be a quadratic Lie superalgebra. An even and symmetric map $D \in \mathrm{End}(\mathfrak{g})$ satisfying $D([X,Y]) = [D(X),Y]$ for all $X,Y \in \mathfrak{g}$ is called a *centromorphism* of $\mathfrak{g}$. The dimension of the space of centromorphisms of $\mathfrak{g}$ is called the *quadratic dimension of* $\mathfrak{g}$ and denoted by $d_q(\mathfrak{g})$. The following Proposition gives the formula of $d_q(\mathfrak{g})$ for reduced quadratic Lie superalgebras with 2-dimensional even part. Its proof goes exactly as the Lie algebra case in Proposition 7.2 [DPU12], the Reader may refer to it.

Proposition 4.20. *Let $\mathfrak{g}$ be a reduced quadratic Lie superalgebra with 2-dimensional even part and $D \in \mathrm{End}(\mathfrak{g})$ be an even symmetric map. Then:*

1. *D is a centromorphism if and only if there exist $\mu \in \mathbb{C}$ and an even symmetric map $\mathsf{Z} : \mathfrak{g} \to \mathcal{Z}(\mathfrak{g})$ such that $\mathsf{Z}|_{[\mathfrak{g},\mathfrak{g}]} = 0$ and $D = \mu\,\mathrm{Id} + \mathsf{Z}$. Moreover D is invertible if and only if $\mu \neq 0$.*
2. $d_q(\mathfrak{g}) = 2 + \dfrac{(\dim(\mathcal{Z}(\mathfrak{g}) - 1))(\dim(\mathcal{Z}(\mathfrak{g}) - 2)}{2}$.

5. Singular quadratic Lie superalgebras of type S_1

Let $\mathfrak{g}$ be a singular quadratic Lie superalgebra of type S_1 such that $\mathfrak{g}_{\bar{0}}$ is non-Abelian. If $[\mathfrak{g}_{\bar{1}}, \mathfrak{g}_{\bar{1}}] = \{0\}$ then $[\mathfrak{g}_{\bar{1}}, \mathfrak{g}] = \{0\}$ and therefore $\mathfrak{g}$ is an orthogonal direct sum of a singular quadratic Lie algebra of type S_1 and a vector space. There is nothing to do. We can assume that $[\mathfrak{g}_{\bar{1}}, \mathfrak{g}_{\bar{1}}] \neq \{0\}$. Fix $\alpha \in \mathcal{V}_I$ and choose $\Omega_0 \in \mathrm{Alt}^2(\mathfrak{g}_{\bar{0}})$, $\Omega_1 \in \mathrm{Sym}^2(\mathfrak{g}_{\bar{1}})$ such that $I = \alpha \wedge \Omega_0 + \alpha \otimes \Omega_1$.

Let $X_{\bar{0}} = \phi^{-1}(\alpha)$ then $X_{\bar{0}} \in \mathcal{Z}(\mathfrak{g})$ and $B(X_{\bar{0}}, X_{\bar{0}}) = 0$. We define linear maps $C_0 : \mathfrak{g}_{\bar{0}} \to \mathfrak{g}_{\bar{0}}$, $C_1 : \mathfrak{g}_{\bar{1}} \to \mathfrak{g}_{\bar{1}}$ by $\Omega_0(X,Y) = B(C_0(X),Y)$ if $X,Y \in \mathfrak{g}_{\bar{0}}$ and $\Omega_1(X,Y) = B(C_1(X),Y)$ if $X,Y \in \mathfrak{g}_{\bar{1}}$. Let $C : \mathfrak{g} \to \mathfrak{g}$ defined by $C(X+Y) = C_0(X) + C_1(Y)$, for all $X \in \mathfrak{g}_{\bar{0}}$, $Y \in \mathfrak{g}_{\bar{1}}$. The proof of the following Proposition is straightforward.

Proposition 5.1. *For all $X,Y \in \mathfrak{g}$, the Lie super-bracket of $\mathfrak{g}$ is defined by:*

$$[X,Y] = B(X_{\bar{0}}, X)C(Y) - B(X_{\bar{0}}, Y)C(X) + B(C(X),Y)X_{\bar{0}}.$$

Now, we show that $\mathfrak{g}_{\bar{0}}$ is solvable. Consider the quadratic Lie algebra $\mathfrak{g}_{\bar{0}}$ with 3-form $I_0 = \alpha \wedge \Omega_0$. Write $\Omega_0 = \sum_{i<j} a_{ij} \alpha_i \wedge \alpha_j$, with $a_{ij} \in \mathbb{C}$. Set $X_i = \phi^{-1}(\alpha_i)$ then

$$C_0 = \sum_{i<j} a_{ij}(\alpha_i \otimes X_j - \alpha_j \otimes X_i).$$

Define the space $W_{I_0} \subset \mathfrak{g}_{\bar 0}^*$ by $\mathcal{W}_{I_0} = \{\iota_{X\wedge Y}(I_0) \mid X,Y \in \mathfrak{g}_{\bar 0}\}$. Then $\mathcal{W}_{I_0} = \phi([\mathfrak{g}_{\bar 0},\mathfrak{g}_{\bar 0}])$ and that implies $\mathrm{Im}(C_0) \subset [\mathfrak{g}_{\bar 0},\mathfrak{g}_{\bar 0}]$. In Section 2, it is known that $\{\alpha, I_0\} = 0$ and then $[X_{\bar 0},\mathfrak{g}_{\bar 0}] = 0$. As a sequence, $B(X_{\bar 0},[\mathfrak{g}_{\bar 0},\mathfrak{g}_{\bar 0}]) = 0$. That deduces $B(X_{\bar 0},\mathrm{Im}(C_0)) = 0$. Therefore $[[\mathfrak{g}_{\bar 0},\mathfrak{g}_{\bar 0}],[\mathfrak{g}_{\bar 0},\mathfrak{g}_{\bar 0}]] = [\mathrm{Im}(C_0),\mathrm{Im}(C_0)] \subset \mathbb{C}X_{\bar 0} \subset \mathcal{Z}(\mathfrak{g})$ and we conclude that $\mathfrak{g}_{\bar 0}$ is solvable.

By B nondegenerate there is an element $Y_{\bar 0} \in \mathfrak{g}_{\bar 0}$ isotropic such that $B(X_{\bar 0},Y_{\bar 0}) = 1$. Moreover, combined with $\mathfrak{g}_{\bar 0}$ a solvable singular quadratic Lie algebra, we can choose $Y_{\bar 0}$ satisfying $C_0(Y_{\bar 0}) = 0$ and we obtain then a consequence as follows:

Corollary 5.2.

1. $C = \mathrm{ad}(Y_{\bar 0})$, $\ker(C) = \mathcal{Z}(\mathfrak{g}) \oplus \mathbb{C}Y_{\bar 0}$ *and* $[\mathfrak{g},\mathfrak{g}] = \mathrm{Im}(C) \oplus \mathbb{C}X_{\bar 0}$.
2. *The Lie superalgebra* $\mathfrak{g}$ *is solvable. Moreover,* $\mathfrak{g}$ *is nilpotent if and only if* C *is nilpotent.*

5.1. Singular quadratic Lie superalgebras of type S_1 *and double extensions*

The description of the Lie super-bracket in Proposition 5.1 allows us to propose a definition of double extension of a quadratic $\mathbb{Z}_2$-graded vector space as follows:

Definition 5.3. Let $(\mathfrak{q} = \mathfrak{q}_{\bar 0} \oplus \mathfrak{q}_{\bar 1}, B_{\mathfrak{q}})$ be a quadratic $\mathbb{Z}_2$-graded vector space and $\overline{C}$ be an even endomorphism of $\mathfrak{q}$. Assume that $\overline{C}$ is skew-supersymmetric, that is, $B(\overline{C}(X),Y) = -B(X,\overline{C}(Y))$, for all $X,Y \in \mathfrak{q}$. Let $(\mathfrak{t} = \mathrm{span}\{X_{\bar 0},Y_{\bar 0}\}, B_{\mathfrak{t}})$ be a 2-dimensional quadratic vector space with the symmetric bilinear form $B_{\mathfrak{t}}$ defined by $B_{\mathfrak{t}}(X_{\bar 0},X_{\bar 0}) = B_{\mathfrak{t}}(Y_{\bar 0},Y_{\bar 0}) = 0$ and $B_{\mathfrak{t}}(X_{\bar 0},Y_{\bar 0}) = 1$.

Consider the vector space $\mathfrak{g} = \mathfrak{t} \overset{\perp}{\oplus} \mathfrak{q}$ equipped with the bilinear form $B = B_{\mathfrak{t}} + B_{\mathfrak{q}}$ and define on $\mathfrak{g}$ the following bracket:

$$[\lambda X_{\bar 0} + \mu Y_{\bar 0} + X, \lambda' X_{\bar 0} + \mu' Y_{\bar 0} + Y] = \mu\overline{C}(Y) - \mu'\overline{C}(X) + B(\overline{C}(X),Y)X_{\bar 0},$$

for all $X,Y \in \mathfrak{q}, \lambda,\mu,\lambda',\mu' \in \mathbb{C}$. Then $(\mathfrak{g},B)$ is a quadratic solvable Lie superalgebra with $\mathfrak{g}_{\bar 0} = \mathfrak{t} \oplus \mathfrak{q}_{\bar 0}$ and $\mathfrak{g}_{\bar 1} = \mathfrak{q}_{\bar 1}$. We say $\mathfrak{g}$ the *double extension of* $\mathfrak{q}$ *by* $\overline{C}$.

Note that an even skew-supersymmetric endomorphism $\overline{C}$ on $\mathfrak{q}$ can be written by $\overline{C} = \overline{C}_0 + \overline{C}_1$ where $\overline{C}_0 \in \mathfrak{o}(\mathfrak{q}_{\bar 0})$ and $\overline{C}_1 \in \mathfrak{sp}(\mathfrak{q}_{\bar 1})$.

Corollary 5.4. *Let* $\mathfrak{g}$ *be the double extension of* $\mathfrak{q}$ *by* $\overline{C}$. *Denote by* $C = \mathrm{ad}(Y_{\bar 0})$ *then one has*

1. $[X,Y] = B(X_{\bar 0},X)C(Y) - B(X_{\bar 0},Y)C(X) + B(C(X),Y)X_{\bar 0}$, *for all* $X,Y \in \mathfrak{g}$.
2. *The Lie superalgebra* $\mathfrak{g}$ *is singular. If* $\overline{C}|_{\mathfrak{q}_{\bar 1}}$ *is nonzero then* $\mathfrak{g}$ *is of type* S_1.

Proof. The assertion (1) is direct from the above definition. Let $\alpha = \phi(X_{\bar{0}})$ and define the bilinear form $\Omega : \mathfrak{g} \to \mathfrak{g}$ by $\Omega(X,Y) = B(C(X),Y)$ for all $X,Y \in \mathfrak{g}$. By B even and supersymmetric, C even and skew-supersymmetric (with respect to B) then $\Omega = \Omega_0 + \Omega_1 \in \mathrm{Alt}^2(\mathfrak{g}_{\bar{0}}) \oplus \mathrm{Sym}^2(\mathfrak{g}_{\bar{1}})$. The formula in (1) can be replaced by $I = \alpha \wedge \Omega_0 + \alpha \otimes \Omega_1 = \alpha \wedge \Omega$. Therefore, $\mathrm{dup}(\mathfrak{g}) \geq 1$ and $\mathfrak{g}$ is singular. If $\overline{C}|_{\mathfrak{q}_{\bar{1}}}$ is nonzero then $\Omega_1 \neq 0$. In this case, $[\mathfrak{g}_{\bar{1}}, \mathfrak{g}_{\bar{1}}] \neq \{0\}$ and thus $\mathrm{dup}(\mathfrak{g}) = 1$. □

As a consequence of Proposition 5.1 and Definition 5.3, one has

Lemma 5.5. *Let $(\mathfrak{g},B)$ be a singular quadratic Lie superalgebra of type S_1. Keep the notations as in Proposition 5.1 and Corollary 5.2. Then $(\mathfrak{g},B)$ is the double extension of $\mathfrak{q} = (\mathbb{C}X_{\bar{0}} \oplus \mathbb{C}Y_{\bar{0}})^\perp$ by $\overline{C} = C|_{\mathfrak{q}}$.*

Remark 5.6. The above definition is a generalization of the definition of double extension of a quadratic vector space by a skew-symmetric map in [DPU12] and Definition 4.6. Moreover, if let $\mathfrak{g} = (\mathbb{C}X_{\bar{0}} \oplus \mathbb{C}Y_{\bar{0}}) \overset{\perp}{\oplus} (\mathfrak{q}_{\bar{0}} \oplus \mathfrak{q}_{\bar{1}})$ be the double extension of $\mathfrak{q} = \mathfrak{q}_{\bar{0}} \oplus \mathfrak{q}_{\bar{1}}$ by $\overline{C} = \overline{C}_0 + \overline{C}_1$ then $\mathfrak{g}_{\bar{0}}$ is the double extension of $\mathfrak{q}_{\bar{0}}$ by $\overline{C}_0$ and the subalgebra $(\mathbb{C}X_{\bar{0}} \oplus \mathbb{C}Y_{\bar{0}}) \overset{\perp}{\oplus} \mathfrak{q}_{\bar{1}}$ is the double extension of $\mathfrak{q}_{\bar{1}}$ by $\overline{C}_1$.

The proof of the Proposition below is completely analogous to the proof of Proposition 4.10, so we omit it.

Proposition 5.7. *Let $\mathfrak{g} = (\mathbb{C}X_{\bar{0}} \oplus \mathbb{C}Y_{\bar{0}}) \overset{\perp}{\oplus} (\mathfrak{q}_{\bar{0}} \oplus \mathfrak{q}_{\bar{1}})$ and $\mathfrak{g}' = (\mathbb{C}X'_{\bar{0}} \oplus \mathbb{C}Y'_{\bar{0}}) \overset{\perp}{\oplus} (\mathfrak{q}_{\bar{0}} \oplus \mathfrak{q}_{\bar{1}})$ be two double extensions of $\mathfrak{q} = \mathfrak{q}_{\bar{0}} \oplus \mathfrak{q}_{\bar{1}}$ by $\overline{C} = \overline{C}_0 + \overline{C}_1$ and $\overline{C'} = \overline{C'_0} + \overline{C'_1}$, respectively. Assume that $\overline{C}_1$ is nonzero. Then*

1. *there exists a Lie superalgebra isomorphism between $\mathfrak{g}$ and $\mathfrak{g}'$ if and only if there exist invertible maps $P \in \mathrm{End}(\mathfrak{q}_{\bar{0}})$, $Q \in \mathrm{End}(\mathfrak{q}_{\bar{1}})$ and a nonzero $\lambda \in \mathbb{C}$ such that*
   (i) $\overline{C'_0} = \lambda P \overline{C}_0 P^{-1}$ and $P^* P \overline{C}_0 = \overline{C}_0$.
   (ii) $\overline{C'_1} = \lambda Q \overline{C}_1 Q^{-1}$ and $Q^* Q \overline{C}_1 = \overline{C}_1$.

   where P^* and Q^* are the adjoint maps of P and Q with respect to $B|_{\mathfrak{q}_{\bar{0}} \times \mathfrak{q}_{\bar{0}}}$ and $B|_{\mathfrak{q}_{\bar{1}} \times \mathfrak{q}_{\bar{1}}}$.
2. *there exists an i-isomorphism between $\mathfrak{g}$ and $\mathfrak{g}'$ if and only if there is a nonzero $\lambda \in \mathbb{C}$ such that $\overline{C'_0}$ is in the $\mathrm{O}(\mathfrak{q}_{\bar{0}})$-adjoint orbit through $\lambda \overline{C}_0$ and $\overline{C'_1}$ is in the $\mathrm{Sp}(\mathfrak{q}_{\bar{1}})$-adjoint orbit through $\lambda \overline{C}_1$.*

Remark 5.8. If let $M = P + Q$ then $M^{-1} = P^{-1} + Q^{-1}$ and $M^* = P^* + Q^*$. The formulas in Proposition 5.7 (1) can be written by $\overline{C'} = \lambda M \overline{C} M^{-1}$ and $M^* M \overline{C} = \overline{C}$. Hence, the classification problem of singular quadratic Lie superalgebras of type

S_1 (up to i-isomorphism) can be reduced to the classification of $\mathrm{O}(\mathfrak{q}_{\bar{0}}) \times \mathrm{Sp}(\mathfrak{q}_{\bar{1}})$-orbits of $\mathfrak{o}(\mathfrak{q}_{\bar{0}}) \oplus \mathfrak{sp}(\mathfrak{q}_{\bar{1}})$, where $\mathrm{O}(\mathfrak{q}_{\bar{0}}) \times \mathrm{Sp}(\mathfrak{q}_{\bar{1}})$ denotes the direct product of two groups $\mathrm{O}(\mathfrak{q}_{\bar{0}})$ and $\mathrm{Sp}(\mathfrak{q}_{\bar{1}})$.

Definition 5.9. Let $\mathfrak{q} = \mathfrak{q}_{\bar{0}} \oplus \mathfrak{q}_{\bar{1}}$ be a quadratic $\mathbb{Z}_2$-graded vector space. An even isomorphism $F \in \mathrm{End}(\mathfrak{q})$ is called an *isometry* of $\mathfrak{q}$ if $F|_{\mathfrak{q}_{\bar{0}}}$ and $F|_{\mathfrak{q}_{\bar{1}}}$ are isometries.

To prove the following Corollary, it is enough to follow exactly the same steps as in Corollary 4.11.

Corollary 5.10. *Let $(\mathfrak{g}, B, \overline{C})$ and $(\mathfrak{g}', B', \overline{C'})$ be double extensions respectively of $(\mathfrak{q}, \overline{B})$ and $(\mathfrak{q}', \overline{B'})$ where $\overline{B} = B|_{\mathfrak{q}\times\mathfrak{q}}$ and $\overline{B'} = B'|_{\mathfrak{q}'\times\mathfrak{q}'}$. Write $\mathfrak{g} = (\mathbb{C}X_{\bar{0}} \oplus \mathbb{C}Y_{\bar{0}}) \overset{\perp}{\oplus} \mathfrak{q}$ and $\mathfrak{g}' = (\mathbb{C}X'_{\bar{0}} \oplus \mathbb{C}Y'_{\bar{0}}) \overset{\perp}{\oplus} \mathfrak{q}'$. Then:*

1. *there exists an i-isomorphism between $\mathfrak{g}$ and $\mathfrak{g}'$ if and only if there exists an isometry $\overline{A} : \mathfrak{q} \to \mathfrak{q}'$ such that $\overline{C'} = \lambda\, \overline{A}\, \overline{C}\, \overline{A}^{-1}$, for some nonzero $\lambda \in \mathbb{C}$.*
2. *there exists a Lie superalgebra isomorphism between $\mathfrak{g}$ and $\mathfrak{g}'$ if and only if there exist even invertible maps $\overline{Q} : \mathfrak{q} \to \mathfrak{q}'$ and $\overline{P} \in \mathrm{End}(\mathfrak{q})$ such that*
   - *(i) $\overline{C'} = \lambda\, \overline{Q}\, \overline{C}\, \overline{Q}^{-1}$ for some nonzero $\lambda \in \mathbb{C}$,*
   - *(ii) $\overline{P}^*\, \overline{P}\, \overline{C} = \overline{C}$ and $\overline{Q}\, \overline{P}^{-1}$ is an isometry from $\mathfrak{q}$ onto $\mathfrak{q}'$.*

5.2. Fitting decomposition of a skew-supersymmetric map

We recall the following useful result (see for instance [DPU12]):

Lemma 5.11. *Let $\overline{C}$ and $\overline{C'}$ be nilpotent elements in $\mathfrak{o}(n)$. Then $\overline{C}$ is conjugate to $\lambda\overline{C'}$ modulo $\mathrm{O}(n)$ for some nonzero $\lambda \in \mathbb{C}$ if and only if $\overline{C}$ is conjugate to $\overline{C'}$.*

Remark that the lemma remains valid if we replace $\mathfrak{o}(n)$ by $\mathfrak{sp}(2n)$ and $\mathrm{O}(n)$ by $\mathrm{Sp}(2n)$.

Proposition 5.12. *Let $\mathfrak{g}$ and $\mathfrak{g}'$ be two nilpotent singular quadratic Lie superalgebras. Then $\mathfrak{g}$ and $\mathfrak{g}'$ are isomorphic if and only if they are i-isomorphic.*

Proof. Singular quadratic Lie superalgebras $\mathfrak{g}$ and $\mathfrak{g}'$ are regarded as double extensions $\mathfrak{g} = (\mathbb{C}X_{\bar{0}} \oplus \mathbb{C}Y_{\bar{0}}) \overset{\perp}{\oplus} \mathfrak{q}$ and $(\mathbb{C}X'_{\bar{0}} \oplus \mathbb{C}Y'_{\bar{0}}) \overset{\perp}{\oplus} \mathfrak{q}'$ by $\overline{C}$ and $\overline{C'}$ where $\mathfrak{q} = \mathfrak{q}_{\bar{0}} \oplus \mathfrak{q}_{\bar{1}}$ and $\mathfrak{q}' = \mathfrak{q}'_{\bar{0}} \oplus \mathfrak{q}'_{\bar{1}}$. By Corollary 5.2, $\overline{C}$ and $\overline{C'}$ are nilpotent. Rewrite $\overline{C} = \overline{C}_0 + \overline{C}_1$ and $\overline{C'} = \overline{C'_0} + \overline{C'_1}$, where $\overline{C}_0 \in \mathfrak{o}(\mathfrak{q}_{\bar{0}})$, $\overline{C'_0} \in \mathfrak{o}(\mathfrak{q}'_{\bar{0}})$, $\overline{C}_1 \in \mathfrak{sp}(\mathfrak{q})$ and $\overline{C'_1} \in \mathfrak{sp}(\mathfrak{q}')$.

If $\mathfrak{g}$ and $\mathfrak{g}'$ are isomorphic then $\dim(\mathfrak{q}_{\bar{0}}) = \dim(\mathfrak{q}'_{\bar{0}})$ and $\dim(\mathfrak{q}_{\bar{1}}) = \dim(\mathfrak{q}'_{\bar{1}})$. Thus, there exist isometries $\overline{F}_0 : \mathfrak{q}'_{\bar{0}} \to \mathfrak{q}_{\bar{0}}$ and $\overline{F}_1 : \mathfrak{q}'_{\bar{1}} \to \mathfrak{q}_{\bar{1}}$ and then we define an isometry $\overline{F} : \mathfrak{q}' \to \mathfrak{q}$ by $\overline{F}(X' + Y') = \overline{F}_0(X') + \overline{F}'_0(Y')$ for all $X' \in \mathfrak{q}'_{\bar{0}}$ and $Y' \in \mathfrak{q}'_{\bar{1}}$.

Set $F:\mathfrak{g}'\to\mathfrak{g}$ by $F(X'_{\bar 0})=X_{\bar 0}$, $F(Y'_{\bar 0})=Y_{\bar 0}$, $F|_{\mathfrak{q}'}=\overline{F}$ and a new Lie super-bracket on $\mathfrak{g}$ by:

$$[X,Y]''=F\left([F^{-1}(X),F^{-1}(Y)]'\right),\ \forall X,Y\in\mathfrak{g}.$$

Denote by $\mathfrak{g}''$ this new quadratic Lie superalgebras. It is easy to see that $\mathfrak{g}''=(\mathbb{C}X_{\bar 0}\oplus\mathbb{C}Y_{\bar 0})\overset{\perp}{\oplus}\mathfrak{q}$ is the double extension of $\mathfrak{q}$ by $\overline{C''}=\overline{F}\ \overline{C'}\ \overline{F}^{-1}$ and $\mathfrak{g}''$ is i-isomorphic to $\mathfrak{g}'$. It need to prove that $\mathfrak{g}''$ is i-isomorphic to $\mathfrak{g}$. Write $\overline{C''}=\overline{C''_0}+\overline{C''_1}\in\mathfrak{o}(\mathfrak{q}_{\bar 0})\oplus\mathfrak{sp}(\mathfrak{q}_{\bar 1})$. Since $\mathfrak{g}$ and $\mathfrak{g}''$ are isomorphic then there exist invertible maps $P:\mathfrak{q}_{\bar 0}\to\mathfrak{q}_{\bar 0}$ and $Q:\mathfrak{q}_{\bar 1}\to\mathfrak{q}_{\bar 1}$ such that $\overline{C''_0}=\lambda\overline{P}\,\overline{C}_0\,\overline{P}^{-1}$ and $\overline{C''_1}=\lambda\overline{Q}\,\overline{C}_1\,\overline{Q}^{-1}$ for some nonzero $\lambda\in\mathbb{C}$. By Lemma 5.11, $\overline{C}_0$ and $\overline{C''_0}$ are conjugate under $\mathrm{O}(\mathfrak{q}_{\bar 0})$, $\overline{C}_1$ and $\overline{C''_1}$ are conjugate under $\mathrm{Sp}(\mathfrak{q}_{\bar 1})$ and we can assume that $\lambda=1$. Therefore $\mathfrak{g}$ and $\mathfrak{g}''$ are i-isomorphic. The Proposition is proved. $\square$

Let now $\mathfrak{g}$ be a singular quadratic Lie superalgebra of type S_1. Write $\mathfrak{g}$ as a double extension of $(\mathfrak{q}=\mathfrak{q}_{\bar 0}\oplus\mathfrak{q}_{\bar 1},\overline{B})$ by $\overline{C}=\overline{C}_0+\overline{C}_1$ where $\overline{C}=\mathrm{ad}(Y_{\bar 0})|_{\mathfrak{q}}$, $\overline{C}_0=\overline{C}|_{\mathfrak{q}_{\bar 0}}$ and $\overline{C}_1=\overline{C}|_{\mathfrak{q}_{\bar 1}}$. We consider the Fitting decomposition of $\overline{C}_0$ on $\mathfrak{q}_{\bar 0}$ and $\overline{C}_1$ on $\mathfrak{q}_{\bar 1}$ by:

$$\mathfrak{q}_{\bar 0}=\mathfrak{q}_{\bar 0}^N\oplus\mathfrak{q}_{\bar 0}^I\ \text{ and }\ \mathfrak{q}_{\bar 1}=\mathfrak{q}_{\bar 1}^N\oplus\mathfrak{q}_{\bar 1}^I$$

where $\mathfrak{q}_{\bar 0}^N$ and $\mathfrak{q}_{\bar 0}^I$ (resp. $\mathfrak{q}_{\bar 1}^N$ and $\mathfrak{q}_{\bar 1}^I$) are $\overline{C}_0$-stable (resp. $\overline{C}_1$-stable), $\overline{C}_0^N=\overline{C}_0|_{\mathfrak{q}_{\bar 0}^N}$ and $\overline{C}_1^N=\overline{C}_1|_{\mathfrak{q}_{\bar 1}^N}$ are nilpotent, $\overline{C}_0^I=\overline{C}_0|_{\mathfrak{q}_{\bar 0}^I}$ and $\overline{C}_1^I=\overline{C}_1|_{\mathfrak{q}_{\bar 1}^I}$ are invertible. Recall that $\overline{C}$ is skew-supersymmetric then $\mathfrak{q}_{\bar 0}^I=(\mathfrak{q}_{\bar 0}^N)^\perp$ in $\mathfrak{g}_{\bar 0}$ and $\mathfrak{q}_{\bar 1}^I=(\mathfrak{q}_{\bar 1}^N)^\perp$ in $\mathfrak{g}_{\bar 1}$.

Next, we set $\mathfrak{q}_N=\mathfrak{q}_{\bar 0}^N\oplus\mathfrak{q}_{\bar 1}^N$ and $\mathfrak{q}_I=\mathfrak{q}_{\bar 0}^I\oplus\mathfrak{q}_{\bar 1}^I$. As a consequence, $\overline{C}_N=\overline{C}|_{\mathfrak{q}_N}$ is nilpotent, $\overline{C}_I=\overline{C}|_{\mathfrak{q}_I}$ is invertible, $[\mathfrak{q}_N,\mathfrak{q}_I]=\{0\}$, the restrictions $\overline{B}_N=\overline{B}|_{\mathfrak{q}_N\times\mathfrak{q}_N}$ and $\overline{B}_I=\overline{B}|_{\mathfrak{q}_I\times\mathfrak{q}_I}$ are nondegenerate and supersymmetric. It is easy to check that $\overline{C}_N=\overline{C}_0^N+\overline{C}_1^N$, $\overline{C}_I=\overline{C}_0^I+\overline{C}_1^I$. Moreover, $\overline{C}_N,\overline{C}_I$ are skew-supersymmetric and they are Fitting components of $\overline{C}$ in $\mathfrak{q}$. Let $\mathfrak{g}_N=(\mathbb{C}X_{\bar 0}\oplus\mathbb{C}Y_{\bar 0})\overset{\perp}{\oplus}\mathfrak{q}_N$ and $\mathfrak{g}_I=(\mathbb{C}X_{\bar 0}\oplus\mathbb{C}Y_{\bar 0})\overset{\perp}{\oplus}\mathfrak{q}_I$. Then $\mathfrak{g}_N$ and $\mathfrak{g}_I$ are subalgebras of $\mathfrak{g}$, $\mathfrak{g}_N$ is the double extension of $\mathfrak{q}_N$ by $\overline{C}_N$, $\mathfrak{g}_I$ is the double extension of $\mathfrak{q}_I$ by $\overline{C}_I$ and $\mathfrak{g}_N$ is a nilpotent singular quadratic Lie superalgebra.

Definition 5.13. The subalgebras $\mathfrak{g}_N$ and $\mathfrak{g}_I$ as above are respectively the *nilpotent* and *invertible Fitting components* of $\mathfrak{g}$.

Definition 5.14. A double extension is called an *invertible* quadratic Lie superalgebra if the corresponding skew-supersymmetric map is invertible.

It is easy to check that the dimension of an invertible quadratic Lie superalgebra must be even. Moreover, by Corollary 5.10, two invertible quadratic Lie superalgebras are isomorphic if and only if they are i-isomorphic. This property still holds for singular quadratic Lie superalgebras of type S_1.

The proof of the Proposition follows Proposition 6.4 in [DPU12]:

Proposition 5.15. *Let $\mathfrak{g}$ and $\mathfrak{g}'$ be singular quadratic Lie superalgebras of type S_1 and $\mathfrak{g}_N$, $\mathfrak{g}_I$, $\mathfrak{g}'_N$, $\mathfrak{g}'_I$ be their Fitting components, respectively. Then*

1. $\mathfrak{g} \overset{\mathrm{i}}{\simeq} \mathfrak{g}'$ *if and only if* $\mathfrak{g}_N \overset{\mathrm{i}}{\simeq} \mathfrak{g}'_N$ *and* $\mathfrak{g}_I \overset{\mathrm{i}}{\simeq} \mathfrak{g}'_I$. *The result remains valid if we replace* $\overset{\mathrm{i}}{\simeq}$ *by* $\simeq$.
2. $\mathfrak{g}$ *and* $\mathfrak{g}'$ *are isomorphic if and only if they are i-isomorphic.*

To prove the main result of this Section, we need the following lemma.

Lemma 5.16. *Let $\mathfrak{g}$ be a reduced singular quadratic Lie superalgebras of type S_1 with $[\mathfrak{g}_{\bar{1}},\mathfrak{g}_{\bar{1}}] \neq 0$ and $D \in \mathrm{End}(\mathfrak{g})$ be an even symmetric map. Then D is a centromorphism if and only if there exist $\mu \in \mathbb{C}$ and an even symmetric map $\mathsf{Z} : \mathfrak{g} \to \mathcal{Z}(\mathfrak{g})$ such that $\mathsf{Z}|_{[\mathfrak{g},\mathfrak{g}]} = 0$ and $D = \mu \,\mathrm{Id} + \mathsf{Z}$. Moreover D is invertible if and only if $\mu \neq 0$.*

Proof. First, $\mathfrak{g}$ can be realized as the double extension $\mathfrak{g} = (\mathbb{C}X_{\bar{0}} \oplus \mathbb{C}Y_{\bar{0}}) \overset{\perp}{\oplus} \mathfrak{q}$ by $C = \mathrm{ad}(Y_{\bar{0}})$ and let $\overline{C} = C|_{\mathfrak{q}}$.

Assume that D is an invertible centromorphism then $D \circ \mathrm{ad}(X) = \mathrm{ad}(X) \circ D$, for all $X \in \mathfrak{g}$ and therefore $DC = CD$. Using formula (1) of Corollary 5.4 and $CD = DC$, from $[D(X),Y_{\bar{0}}] = [X, D(Y_{\bar{0}})]$ we find

$$D(C(X)) = \mu C(X),\ \forall\, X \in \mathfrak{g},\ \text{where } \mu = B(D(X_{\bar{0}}),Y_{\bar{0}}).$$

Since D is invertible, then $\mu \neq 0$ and $C(D - \mu\,\mathrm{Id}) = 0$. Recall that $\ker(C) = \mathbb{C}X_{\bar{0}} \oplus \ker(\overline{C}) \oplus \mathbb{C}Y_{\bar{0}} = \mathcal{Z}(\mathfrak{g}) \oplus \mathbb{C}Y_{\bar{0}}$, there exist a map $\mathsf{Z} : \mathfrak{g} \to \mathcal{Z}(\mathfrak{g})$ and $\varphi \in \mathfrak{g}^*$ such that $D - \mu\,\mathrm{Id} = \mathsf{Z} + \varphi \otimes Y_{\bar{0}}$.

We must show that $\varphi = 0$. Indeed, D maps $[\mathfrak{g},\mathfrak{g}]$ into itself and $Y_{\bar{0}} \notin [\mathfrak{g},\mathfrak{g}]$, so $\varphi|_{[\mathfrak{g},\mathfrak{g}]} = 0$. One has $[\mathfrak{g},\mathfrak{g}] = \mathbb{C}X_{\bar{0}} \oplus \mathrm{Im}(\overline{C})$. If $X \in \mathrm{Im}(\overline{C})$, let $X = C(Y)$. Then $D(X) = D(C(Y)) = \mu C(Y)$, so $D(X) = \mu X$. For $Y_{\bar{0}}$, $D([Y_{\bar{0}},X]) = DC(X) = \mu C(X)$ for all $X \in \mathfrak{g}$. But also, $D([Y_{\bar{0}},X]) = [D(Y_{\bar{0}}),X] = \mu C(X) + \varphi(Y_{\bar{0}})C(X)$, hence $\varphi(Y_{\bar{0}}) = 0$. As a consequence, $D(Y_{\bar{0}}) = \mu Y_{\bar{0}} + \mathsf{Z}(Y_{\bar{0}})$.

Now, we prove that $D(X_{\bar{0}}) = \mu X_{\bar{0}}$. Indeed, since D is even and $[\mathfrak{g}_{\bar{1}},\mathfrak{g}_{\bar{1}}] = \mathbb{C}X_{\bar{0}}$ then one has $D(X_{\bar{0}}) \subset D([\mathfrak{g}_{\bar{1}},\mathfrak{g}_{\bar{1}}]) = [D(\mathfrak{g}_{\bar{1}}),\mathfrak{g}_{\bar{1}}] \subset [\mathfrak{g}_{\bar{1}},\mathfrak{g}_{\bar{1}}] = \mathbb{C}X_{\bar{0}}$. It implies $D(X_{\bar{0}}) = aX_{\bar{0}}$. Combined with $B(D(Y_{\bar{0}}),X_{\bar{0}}) = B(Y_{\bar{0}},D(X_{\bar{0}}))$, we obtain $\mu = a$.

Let $X \in \mathfrak{q}$, $B(D(X_{\bar{0}}),X) = \mu B(X_{\bar{0}},X) = 0$. Moreover, $B(D(X_{\bar{0}}),X) = B(X_{\bar{0}},D(X))$, so $\varphi(X) = 0$.

By [Ben03], $\mathcal{C}(\mathfrak{g})$ is generated by invertible centromorphisms then the necessary condition of Lemma is finished. The sufficiency is obvious. $\square$

Proposition 5.17. *The* dup*-number is invariant under Lie superalgebra isomorphisms, i.e. if* $(\mathfrak{g},B)$ *and* $(\mathfrak{g}',B')$ *are quadratic Lie superalgebras with* $\mathfrak{g}\simeq\mathfrak{g}'$*, then* $\mathrm{dup}(\mathfrak{g})=\mathrm{dup}(\mathfrak{g}')$.

Proof. By Lemma 2.4 we can assume $\mathfrak{g}$ reduced. By Proposition 2.3, $\mathfrak{g}'$ is also reduced. Since $\mathfrak{g}\simeq\mathfrak{g}'$ then we can identify $\mathfrak{g}=\mathfrak{g}'$ as a Lie superalgebra equipped with the bilinear forms B, B' and we have two dup-numbers: $\mathrm{dup}_B(\mathfrak{g})$ and $\mathrm{dup}_{B'}(\mathfrak{g})$.

We start with the case $\mathrm{dup}_B(\mathfrak{g})=3$. Since $\mathfrak{g}$ is reduced then $\mathfrak{g}_{\bar 1}=\{0\}$ and $\mathfrak{g}$ is a reduced singular quadratic Lie algebra of type S_3. By [PU07], $\dim([\mathfrak{g},\mathfrak{g}])=3$ and then $\mathrm{dup}_{B'}(\mathfrak{g})=3$.

If $\mathrm{dup}_B(\mathfrak{g})=1$, then $\mathfrak{g}$ is of type S_1 with respect to B. There are two cases: $[\mathfrak{g}_{\bar 1},\mathfrak{g}_{\bar 1}]=\{0\}$ and $[\mathfrak{g}_{\bar 1},\mathfrak{g}_{\bar 1}]\neq\{0\}$. If $[\mathfrak{g}_{\bar 1},\mathfrak{g}_{\bar 1}]=0$ then $\mathfrak{g}_{\bar 1}=\{0\}$ by $\mathfrak{g}$ reduced. In this case, $\mathfrak{g}$ is a reduced singular quadratic Lie algebra of type S_1. By [DPU12], $\mathfrak{g}$ is also of type S_1 with the bilinear form B', i.e. $\mathrm{dup}_{B'}(\mathfrak{g})=1$.

Assume that $[\mathfrak{g}_{\bar 1},\mathfrak{g}_{\bar 1}]\neq\{0\}$. By the previous lemma, the bilinear form B' defines an associated invertible centromorphism $D=\mu\,\mathrm{Id}+\mathsf{Z}$ for some nonzero $\mu\in\mathbb{C}$ and $\mathsf{Z}:\mathfrak{g}\to\mathscr{Z}(\mathfrak{g})$ satisfying $\mathsf{Z}|_{[\mathfrak{g},\mathfrak{g}]}=0$. For all $X,Y,Z\in\mathfrak{g}$, one has:

$$I'(X,Y,Z)=B'([X,Y],Z)=B(D([X,Y]),Z)=B([D(X),Y],Z)=\mu B([X,Y],Z).$$

That means $I'=\mu I$ and then $\mathrm{dup}_{B'}(\mathfrak{g})=\mathrm{dup}_B(\mathfrak{g})=1$.

Finally, if $\mathrm{dup}_B(\mathfrak{g})=0$ then $\mathfrak{g}$ cannot be of type S_3 or S_1 with respect to B', so $\mathrm{dup}_{B'}(\mathfrak{g})=0$. □

Proposition 5.18. *Let* $\mathfrak{g}$ *be a reduced singular quadratic Lie superalgebra of type* S_1 *with* $[\mathfrak{g}_{\bar 1},\mathfrak{g}_{\bar 1}]\neq 0$*. Then the quadratic dimension of* $\mathfrak{g}$ *can be calculated as follows:*

$$d_q(\mathfrak{g})=1+\frac{\dim(\mathscr{Z}(\mathfrak{g})_{\bar 0})(1+\dim(\mathscr{Z}(\mathfrak{g})_{\bar 0}))}{2}+\frac{\dim(\mathscr{Z}(\mathfrak{g})_{\bar 1})(\dim(\mathscr{Z}(\mathfrak{g})_{\bar 1})-1)}{2}.$$

Proof. Keep the notation as in Lemma 5.16. We set $\mathscr{Z}(\mathfrak{g})_{\bar 0}=\mathscr{Z}(\mathfrak{g})\cap\mathfrak{g}_{\bar 0}$, $\mathscr{Z}(\mathfrak{g})_{\bar 1}=\mathscr{Z}(\mathfrak{g})\cap\mathfrak{g}_{\bar 1}$, $[\mathfrak{g},\mathfrak{g}]_{\bar 0}=[\mathfrak{g},\mathfrak{g}]\cap\mathfrak{g}_{\bar 0}$ and $[\mathfrak{g},\mathfrak{g}]_{\bar 1}=[\mathfrak{g},\mathfrak{g}]\cap\mathfrak{g}_{\bar 1}$. It is obvious that $X_{\bar 0}\in\mathscr{Z}(\mathfrak{g})_{\bar 0}\subset[\mathfrak{g},\mathfrak{g}]_{\bar 0}$ and $\mathscr{Z}(\mathfrak{g})_{\bar 1}\subset[\mathfrak{g},\mathfrak{g}]_{\bar 1}$. In other words, $\mathscr{Z}(\mathfrak{g})_{\bar 0}$ and $\mathscr{Z}(\mathfrak{g})_{\bar 1}$ are totally isotropic subspaces of $\mathfrak{g}_{\bar 0}$ and $\mathfrak{g}_{\bar 1}$, respectively. Rewrite $\mathscr{Z}(\mathfrak{g})_{\bar 0}=\mathbb{C}X_{\bar 0}\oplus\mathfrak{l}_{\bar 0}$. Then there exist totally isotropic subspaces $\mathfrak{u}_{\bar 0}\oplus\mathbb{C}Y_{\bar 0}$ of $\mathfrak{g}_{\bar 0}$ and $\mathfrak{u}_{\bar 1}$ of $\mathfrak{g}_{\bar 1}$ such that $\mathfrak{g}_{\bar 0}=[\mathfrak{g},\mathfrak{g}]_{\bar 0}\oplus(\mathfrak{u}_{\bar 0}\oplus\mathbb{C}Y_{\bar 0})$, $\mathfrak{g}_{\bar 1}=[\mathfrak{g},\mathfrak{g}]_{\bar 1}\oplus\mathfrak{u}_{\bar 1}$, the subspaces $\mathscr{Z}(\mathfrak{g})_{\bar 0}\oplus(\mathfrak{u}_{\bar 0}\oplus\mathbb{C}Y_{\bar 0})$ and $\mathscr{Z}(\mathfrak{g})_{\bar 1}\oplus\mathfrak{u}_{\bar 1}$ are nondegenerate. Let us define $\mathsf{Z}:\mathfrak{u}_{\bar 0}\oplus\mathbb{C}Y_{\bar 0}\oplus\mathfrak{u}_{\bar 1}\to\mathfrak{l}_{\bar 0}\oplus\mathbb{C}X_{\bar 0}\oplus\mathscr{Z}(\mathfrak{g})_{\bar 1}$ by setting bases $\{X_1=X_{\bar 0},X_2,\dots,X_r\}$ of $\mathfrak{l}_{\bar 0}\oplus\mathbb{C}X_{\bar 0}$, $\{Y_1,\dots,Y_t\}$ of $\mathscr{Z}(\mathfrak{g})_{\bar 1}$, $\{X'_1=Y_{\bar 0},X'_2,\dots,X'_r\}$ of $\mathfrak{u}_{\bar 0}\oplus\mathbb{C}Y_{\bar 0}$ and $\{Y'_1,\dots,Y'_t\}$ of $\mathfrak{u}_{\bar 1}$ such that $B(X_i,X'_j)=\delta_{ij}$, $B(Y_k,Y'_l)=\delta_{kl}$. Then the map Z is completely defined by

$$\mathsf{Z}\left(\sum_{j=1}^{r}x_jX'_j\right)=\sum_{i=1}^{r}\left(\sum_{j=1}^{r}\mu_{ij}x_j\right)X_i\ \text{ and }\ \mathsf{Z}\left(\sum_{j=1}^{t}y_jY'_j\right)=\sum_{i=1}^{t}\left(\sum_{j=1}^{t}\nu_{ij}y_j\right)Y_i$$

with $\mu_{ij} = \mu_{ji} = B(X'_i, \mathrm{Z}(X'_j))$ and $\nu_{ij} = -\nu_{ji} = B(Y'_i, \mathrm{Z}(Y'_j))$. Therefore, the proposition is proved. $\square$

6. Quasi-singular quadratic Lie superalgebras

By Definition 5.3, it is natural to ask: let $(\mathfrak{q} = \mathfrak{q}_{\bar{0}} \oplus \mathfrak{q}_{\bar{1}}, B_{\mathfrak{q}})$ be a quadratic $\mathbb{Z}_2$-graded vector space and $\overline{C} \in \mathrm{End}(\mathfrak{q})$. Let $(\mathfrak{t} = \mathrm{span}\{X_{\bar{1}}, Y_{\bar{1}}\}, B_{\mathfrak{t}})$ be a 2-dimensional *symplectic* vector space with $B_{\mathfrak{t}}(X_{\bar{1}}, Y_{\bar{1}}) = 1$. Is there an extension $\mathfrak{g} = \mathfrak{q} \overset{\perp}{\oplus} \mathfrak{t}$ such that $\mathfrak{g}$ equipped with the bilinear form $B = B_{\mathfrak{q}} + B_{\mathfrak{t}}$ becomes a quadratic Lie superalgebra such that $\mathfrak{g}_{\bar{0}} = \mathfrak{q}_{\bar{0}}$, $\mathfrak{g}_{\bar{1}} = \mathfrak{q}_{\bar{1}} \oplus \mathfrak{t}$ and the Lie super-bracket is represented by $\overline{C}$? In this section, we will give an affirmative answer to this question.

The dup-number and the form of the associated invariant I in the previous sections suggest that it would be also interesting to study a quadratic Lie superalgebra $\mathfrak{g}$ whose associated invariant I has the form $I = J \wedge p$ where $p \in \mathfrak{g}_{\bar{1}}^*$ is nonzero, $J \in \mathrm{Alt}^1(\mathfrak{g}_{\bar{0}}) \otimes \mathrm{Sym}^1(\mathfrak{g}_{\bar{1}})$ is indecomposable. We obtain the first result as follows:

Proposition 6.1. $\{J,J\} = \{p,J\} = 0$.

Proof. Apply Proposition 1.2 to obtain

$$\{I,I\} = -\{J,J\} \wedge p \wedge p + 2J \wedge \{p,J\} \wedge p - J \wedge J \wedge \{p,p\}.$$

Since $J \wedge J = 0$ and $\{I,I\} = 0$ one has $\{J,J\} \wedge p \wedge p = 2J \wedge \{p,J\} \wedge p$. That means $\{J,J\} \wedge p = 2J \wedge \{p,J\}$.

If $\{J,J\} \neq 0$ then $\{J,J\} \wedge p \neq 0$, so $J \wedge \{p,J\} \neq 0$. Note that $\{p,J\} \in \mathrm{Alt}^1(\mathfrak{g}_{\bar{0}})$ so J must contain the factor p, i.e. $J = \alpha \otimes p$ where $\alpha \in \mathfrak{g}_{\bar{0}}^*$. But $\{p,J\} = \{p, \alpha \otimes p\} = -\alpha \otimes \{p,p\} = 0$ since $\{p,p\} = 0$. This is a contradiction and therefore $\{J,J\} = 0$.

As a consequence, $J \wedge \{p,J\} = 0$. Set $\alpha = \{p,J\} \in \mathrm{Alt}^1(\mathfrak{g}_{\bar{0}})$ then we have $J \wedge \alpha = 0$. If $\alpha \neq 0$ then J must have the form $J = \alpha \otimes q$ where $q \in \mathrm{Sym}^1(\mathfrak{g}_{\bar{1}})$. That is a contradiction since J is indecomposable. $\square$

Definition 6.2. We continue to keep the condition $I = J \wedge p$ with $p \in \mathfrak{g}_{\bar{1}}^*$ nonzero and $J \in \mathrm{Alt}^1(\mathfrak{g}_{\bar{0}}) \otimes \mathrm{Sym}^1(\mathfrak{g}_{\bar{1}})$ indecomposable. We can assume that $J = \sum_{i=1}^{n} \alpha_i \otimes p_i$ where $\alpha_i \in \mathrm{Alt}^1(\mathfrak{g}_{\bar{0}})$, $i = 1, \ldots, n$ are linearly independent and $p_i \in \mathrm{Sym}^1(\mathfrak{g}_{\bar{1}})$. A quadratic Lie superalgebra having such associated invariant I is called a *quasi-singular quadratic Lie superalgebra*.

Let $U = \mathrm{span}\{\alpha_1, \dots, \alpha_n\}$ and $V = \mathrm{span}\{p_1, \dots, p_n\}$, one has $\dim(U)$ and $\dim(V)$ more than 1 by if there is a contrary then J is decomposable. Using Definition 1.1, we have

$$\{J,J\} = -\sum_{i,j=1}^{n} \left(\{\alpha_i, \alpha_j\} \otimes p_i p_j + (\alpha_i \wedge \alpha_j) \otimes \{p_i, p_j\}\right).$$

Since $\{J,J\} = 0$ and $\alpha_i,\ i = 1, \dots, n$ are linearly independent then $\{p_i, p_j\} = 0$, for all i, j. It implies that $\{p_i, J\} = 0$, for all i.

Moreover, since $\{p,J\} = 0$ we obtain $\{p, p_i\} = 0$, consequently $\{p_i, I\} = 0$, for all i and $\{p, I\} = 0$. By Corollary 1.7 and Lemma 1.14 we conclude that $\phi^{-1}(V + \mathbb{C}p)$ is a subspace of $\mathcal{Z}(\mathfrak{g})$ and totally isotropic.

Now, let $\{q_1, \dots, q_m\}$ be a basis of V then J can be rewritten by $J = \sum_{j=1}^{m} \beta_j \otimes q_j$ where $\beta_j \in U$, for all j. One has:

$$\{J,J\} = -\sum_{i,j=1}^{m} \left(\{\beta_i, \beta_j\} \otimes q_i q_j + (\beta_i \wedge \beta_j) \otimes \{q_i, q_j\}\right).$$

By the linear independence of the system $\{q_i q_j\}$, we obtain $\{\beta_i, \beta_j\} = 0$, for all i, j. It implies that $\{\beta_j, I\} = 0$, equivalently $\phi^{-1}(\beta_j) \in \mathcal{Z}(\mathfrak{g})$, for all j. Therefore, we always can begin with $J = \sum_{i=1}^{n} \alpha_i \otimes p_i$ satisfying the following conditions:

(i) $\alpha_i,\ i = 1, \dots, n$ are linearly independent,

(ii) $\phi^{-1}(U)$ and $\phi^{-1}(V + \mathbb{C}p)$ are totally isotropic subspaces of $\mathcal{Z}(\mathfrak{g})$ where $U = \mathrm{span}\{\alpha_1, \dots, \alpha_n\}$ and $V = \mathrm{span}\{p_1, \dots, p_n\}$.

Let $X_{\bar{0}}^i = \phi^{-1}(\alpha_i)$, $X_{\bar{1}}^i = \phi^{-1}(p_i)$, for all i and $C : \mathfrak{g} \to \mathfrak{g}$ defined by

$$J(X,Y) = B(C(X),Y),\ \forall\, X,Y \in \mathfrak{g}.$$

then the proof of the below lemma is lengthy but straightforward, so we omit it.

Lemma 6.3. *The mapping C is a skew-supersymmetric homogeneous endomorphism of odd degree and* $\mathrm{Im}(C) \subset \mathcal{Z}(\mathfrak{g})$. *Recall that if C is a homogeneous endomorphism of degree c of $\mathfrak{g}$ satisfying $B(C(X),Y) = -(-1)^{cx} B(X, C(Y))$ for all $X \in \mathfrak{g}_x$, $Y \in \mathfrak{g}$ then we say C* skew-supersymmetric *(with respect to B).*

Proposition 6.4. *Let $X_{\bar{1}} = \phi^{-1}(p)$ then for all $X \in \mathfrak{g}_{\bar{0}}$, $Y, Z \in \mathfrak{g}_{\bar{1}}$ one has:*

1. $[X,Y] = -B(C(X),Y)X_{\bar{1}} - B(X_{\bar{1}},Y)C(X)$,
2. $[Y,Z] = B(X_{\bar{1}},Y)C(Z) + B(X_{\bar{1}},Z)C(Y)$,

3. $X_{\bar{1}} \in \mathscr{Z}(\mathfrak{g})$ *and* $C(X_{\bar{1}}) = 0$.

Proof. Let $X \in \mathfrak{g}_{\bar{0}}$, $Y,Z \in \mathfrak{g}_{\bar{1}}$ then

$$\begin{aligned} B([X,Y],Z) = J \wedge p(X,Y,Z) &= -J(X,Y)p(Z) - J(X,Z)p(Y) \\ &= -B(C(X),Y)B(X_{\bar{1}},Z) - B(C(X),Z)B(X_{\bar{1}},Y). \end{aligned}$$

Therefore, (1) and (2) follow.

Since $\{p,I\} = 0$ then $X_{\bar{1}} \in \mathscr{Z}(\mathfrak{g})$. Moreover, $\{p,p_i\} = 0$ imply $B(X_{\bar{1}},X_{\bar{1}}^i) = 0$, for all i. It means $B(X_{\bar{1}},\mathrm{Im}(C)) = 0$. And since $B(C(X_{\bar{1}}),X) = B(X_{\bar{1}},C(X)) = 0$, for all $X \in \mathfrak{g}$ then $C(X_{\bar{1}}) = 0$. $\square$

Let W be a complementary subspace of $\mathrm{span}\{X_{\bar{1}}^1,\dots,X_{\bar{1}}^n,X_{\bar{1}}\}$ in $\mathfrak{g}_{\bar{1}}$ and $Y_{\bar{1}}$ be an element in W such that $B(X_{\bar{1}},Y_{\bar{1}}) = 1$. Let $X_{\bar{0}} = C(Y_{\bar{1}})$, $\mathfrak{q} = (\mathbb{C}X_1 \oplus \mathbb{C}Y_{\bar{1}})^{\perp}$ and $B_{\mathfrak{q}} = B|_{\mathfrak{q}\times\mathfrak{q}}$ then we have the following corollary:

Corollary 6.5.

1. $[Y_{\bar{1}},Y_{\bar{1}}] = 2X_{\bar{0}}$*,* $[Y_{\bar{1}},X] = C(X) - B(X,X_{\bar{0}})X_{\bar{1}}$ *and* $[X,Y] = -B(C(X),Y)X_{\bar{1}}$*, for all* $X,Y \in \mathfrak{q} \oplus \mathbb{C}X_{\bar{1}}$.
2. $[\mathfrak{g},\mathfrak{g}] \subset \mathrm{Im}(C) + \mathbb{C}X_{\bar{1}} \subset \mathscr{Z}(\mathfrak{g})$ *so* $\mathfrak{g}$ *is 2-step nilpotent. If* $\mathfrak{g}$ *is reduced then* $[\mathfrak{g},\mathfrak{g}] = \mathrm{Im}(C) + \mathbb{C}X_{\bar{1}} = \mathscr{Z}(\mathfrak{g})$.
3. $C^2 = 0$.

Proof.

1. The assertion (1) is obvious by Proposition 6.4.
2. Note that $X_{\bar{0}} \in \mathrm{Im}(C)$ so $[\mathfrak{g},\mathfrak{g}] \subset \mathrm{Im}(C) + \mathbb{C}X_{\bar{1}}$. By Lemma 6.3 and Proposition 6.4, $\mathrm{Im}(C) + \mathbb{C}X_{\bar{1}} \subset \mathscr{Z}(\mathfrak{g})$. If $\mathfrak{g}$ is reduced then $\mathscr{Z}(\mathfrak{g}) \subset [\mathfrak{g},\mathfrak{g}]$ and therefore $[\mathfrak{g},\mathfrak{g}] = \mathrm{Im}(C) + \mathbb{C}X_{\bar{1}} = \mathscr{Z}(\mathfrak{g})$.
3. By $\mathfrak{g}$ 2-step nilpotent then $0 = [Y_{\bar{1}},2X_{\bar{0}}] = 2C(X_{\bar{0}}) - 2B(X_{\bar{0}},X_{\bar{0}})X_{\bar{1}}$. Since $X_{\bar{0}} = C(Y_{\bar{1}})$ and $\mathrm{Im}(C)$ is totally isotropic then $B(X_{\bar{0}},X_{\bar{0}}) = 0$ and therefore $C(X_{\bar{0}}) = C^2(Y_{\bar{1}}) = 0$.
   If $X \in \mathfrak{q} \oplus \mathbb{C}X_{\bar{1}}$ then $0 = [Y_{\bar{1}},[Y_{\bar{1}},X]] = [Y_{\bar{1}},C(X)]$. By the choice of $Y_{\bar{1}}$, it is sure that $C(X) \in \mathfrak{q} \oplus \mathbb{C}X_{\bar{1}}$. Therefore, one has:

   $$0 = [Y_{\bar{1}},C(X)] = C^2(X) - B(C(X),X_{\bar{0}})X_{\bar{1}} = C^2(X) - B(C(X),C(Y_{\bar{1}}))X_{\bar{1}}.$$

   By $\mathrm{Im}(C)$ totally isotropic, one has $C^2(X) = 0$.

$\square$

Now, we consider a special case: $X_{\bar{0}} = 0$. As a consequence, $[Y_{\bar{1}}, Y_{\bar{1}}] = 0$, $[Y_{\bar{1}}, X] = C(X)$ and $[X,Y] = -B(C(X),Y)X_{\bar{1}}$, for all $X,Y \in \mathfrak{q}$. Let $X \in \mathfrak{q}$ and assume that $C(X) = C_1(X) + aX_1$ where $C_1(X) \in \mathfrak{q}$ then

$$0 = B([Y_{\bar{1}}, Y_{\bar{1}}], X) = B(Y_{\bar{1}}, [Y_{\bar{1}}, X]) = B(Y_{\bar{1}}, C_1(X) + aX_1) = a.$$

It shows that $C(X) \in \mathfrak{q}$, for all $X \in \mathfrak{q}$ and therefore we have an affirmative answer of the above question as follows:

Proposition 6.6. *Let $(\mathfrak{q} = \mathfrak{q}_{\bar{0}} \oplus \mathfrak{q}_{\bar{1}}, B_{\mathfrak{q}})$ be a quadratic $\mathbb{Z}_2$-graded vector space and $\overline{C}$ be an odd endomorphism of $\mathfrak{q}$ such that $\overline{C}$ is skew-supersymmetric and $\overline{C}^2 = 0$. Let $(\mathfrak{t} = \mathrm{span}\{X_{\bar{1}}, Y_{\bar{1}}\}, B_{\mathfrak{t}})$ be a 2-dimensional symplectic vector space with $B_{\mathfrak{t}}(X_{\bar{1}}, Y_{\bar{1}}) =$ 1. Consider the space $\mathfrak{g} = \mathfrak{q} \overset{\perp}{\oplus} \mathfrak{t}$ and define the product on $\mathfrak{g}$ by:*

$$[Y_{\bar{1}}, Y_{\bar{1}}] = [X_{\bar{1}}, \mathfrak{g}] = 0,\ [Y_{\bar{1}}, X] = \overline{C}(X) \ \textit{ and } [X,Y] = -B_{\mathfrak{q}}(\overline{C}(X), Y)X_{\bar{1}}$$

for all $X \in \mathfrak{q}$. Then $\mathfrak{g}$ becomes a 2-nilpotent quadratic Lie superalgebra with the bilinear form $B = B_{\mathfrak{q}} + B_{\mathfrak{t}}$. Moreover, one has $\mathfrak{g}_{\bar{0}} = \mathfrak{q}_{\bar{0}}$, $\mathfrak{g}_{\bar{1}} = \mathfrak{q}_{\bar{1}} \oplus \mathfrak{t}$.

Remark 6.7. The method above remains valid for the elementary quadratic Lie superalgebra $\mathfrak{g}_6^s$ with I decomposable (see Section 3) as follows: let $\mathfrak{q} = (\mathbb{C}X_{\bar{0}} \oplus \mathbb{C}Y_{\bar{0}}) \oplus (\mathbb{C}Z_{\bar{1}} \oplus \mathbb{C}T_{\bar{1}})$ where $\mathfrak{q}_{\bar{0}} = \mathrm{span}\{X_{\bar{0}}, Y_{\bar{0}}\}$, $\mathfrak{q}_{\bar{1}} = \mathrm{span}\{Z_{\bar{1}}, T_{\bar{1}}\}$ and the bilinear form $B_{\mathfrak{q}}$ is defined by $B_{\mathfrak{q}}(X_{\bar{0}}, Y_{\bar{0}}) = B_{\mathfrak{q}}(Z_{\bar{1}}, T_{\bar{1}}) = 1$, the other are zero. Let $C : \mathfrak{q} \to \mathfrak{q}$ be a linear map defined by:

$$C = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Then C is odd and $C^2 = 0$. Set the vector space $\mathfrak{g} = \mathfrak{q} \overset{\perp}{\oplus} \mathfrak{t}$, where $(\mathfrak{t} = \mathrm{span}\{X_{\bar{1}}, Y_{\bar{1}}\}, B_{\mathfrak{t}})$ is a 2-dimensional symplectic vector space with $B_{\mathfrak{t}}(X_{\bar{1}}, Y_{\bar{1}}) = 1$. Then $\mathfrak{g} = \mathfrak{g}_6^s$ with the Lie super-bracket defined as in Proposition 6.6.

It remains to consider $X_{\bar{0}} \neq 0$. The fact is that C may be not stable on $\mathfrak{q}$, that is, $C(X) \in \mathfrak{q} \oplus \mathbb{C}X_{\bar{1}}$ if $X \in \mathfrak{q}$ but that we need here is an action stable on $\mathfrak{q}$. Therefore, we decompose C by $C(X) = \overline{C}(X) + \varphi(X)X_{\bar{1}}$, for all $X \in \mathfrak{q}$ where $\overline{C} : \mathfrak{q} \to \mathfrak{q}$ and $\varphi : \mathfrak{q} \to \mathbb{C}$. Since $B(C(Y_{\bar{1}}), X) = B(Y_{\bar{1}}, C(X)$ then $\varphi(X) = -B(X_{\bar{0}}, X) = -B(X, X_{\bar{0}})$, for all $X \in \mathfrak{q}$. Moreover, C is odd degree on $\mathfrak{g}$ and skew-supersymmetric (with respect to B) implies that $\overline{C}$ is also odd on $\mathfrak{q}$ and skew-supersymmetric (with respect to $B_{\mathfrak{q}}$). It is easy to see that $\overline{C}^2 = 0$, $\overline{C}(X_{\bar{0}}) = 0$ and we have the following result:

Corollary 6.8. *Keep the notations as in Corollary 6.5 and replace $2X_{\bar{0}}$ by $X_{\bar{0}}$ then for all $X, Y \in \mathfrak{q}$, one has:*

$$[Y_{\bar{1}}, Y_{\bar{1}}] = X_{\bar{0}},\ [Y_{\bar{1}}, X] = \overline{C}(X) - B(X, X_{\bar{0}})X_{\bar{1}},\ \text{and } [X, Y] = -B(\overline{C}(X), Y)X_{\bar{1}}.$$

Hence, we have a more general result of Proposition 6.6:

Proposition 6.9. *Let $(\mathfrak{q} = \mathfrak{q}_{\bar{0}} \oplus \mathfrak{q}_{\bar{1}}, B_{\mathfrak{q}})$ be a quadratic $\mathbb{Z}_2$-graded vector space and $\overline{C}$ an odd endomorphism of $\mathfrak{q}$ such that $\overline{C}$ is skew-supersymmetric and $\overline{C}^2 = 0$. Let $X_{\bar{0}}$ be an isotropic element of $\mathfrak{q}_{\bar{0}}$, $X_{\bar{0}} \in \ker(\overline{C})$ and $(\mathfrak{t} = \operatorname{span}\{X_{\bar{1}}, Y_{\bar{1}}\}, B_{\mathfrak{t}})$ be a 2-dimensional symplectic vector space with $B_{\mathfrak{t}}(X_{\bar{1}}, Y_{\bar{1}}) = 1$. Consider the space $\mathfrak{g} = \mathfrak{q} \overset{\perp}{\oplus} \mathfrak{t}$ and define the product on $\mathfrak{g}$ by:*

$$[Y_{\bar{1}}, Y_{\bar{1}}] = X_{\bar{0}},\ [Y_{\bar{1}}, X] = \overline{C}(X) - B_{\mathfrak{q}}(X, X_{\bar{0}})X_{\bar{1}} \text{ and } [X, Y] = -B_{\mathfrak{q}}(\overline{C}(X), Y)X_{\bar{1}}$$

for all $X \in \mathfrak{q}$. Then $\mathfrak{g}$ becomes a 2-nilpotent quadratic Lie superalgebra with the bilinear form $B = B_{\mathfrak{q}} + B_{\mathfrak{t}}$. Moreover, one has $\mathfrak{g}_{\bar{0}} = \mathfrak{q}_{\bar{0}}$, $\mathfrak{g}_{\bar{1}} = \mathfrak{q}_{\bar{1}} \oplus \mathfrak{t}$.

A quadratic Lie superalgebra obtained in the above Proposition is a special case of the generalized double extensions given in [BBB] where the authors consider the generalized double extension of a quadratic $\mathbb{Z}_2$-graded vector space (regarded as an Abelian superalgebra) by a one-dimensional Lie superalgebra.

7. Appendix: Adjoint orbits of $\mathfrak{sp}(2n)$ and $\mathfrak{o}(m)$

This appendix recalls a fundamental and really interesting problem in Lie theory that is necessary for the paper: the classification of adjoint orbits of classical Lie algebras $\mathfrak{sp}(2n)$ and $\mathfrak{o}(m)$ where m, $n \in \mathbb{N}^*$. A brief overview can be found in [Hum95] with interesting discussions. Many results can be found in [CM93].

A different point here is to use the Fitting decomposition to review this problem. In particular, we parametrize the *invertible* component in the Fitting decomposition of a skew-symmetric map and from this, we give an explicit classification for $\mathrm{Sp}(2n)$-adjoint orbits of $\mathfrak{sp}(2n)$ and $\mathrm{O}(m)$-adjoint orbits of $\mathfrak{o}(m)$ in the general case. In other words, we establish a one-to-one correspondence between the set of orbits and some set of indices. This is an rather obvious and classical result but in our knowledge there is not a reference for that mentioned before.

Let V be a m-dimensional complex vector space endowed with a nondegenerate bilinear form B_ε where $\varepsilon = \pm 1$ such that $B_\varepsilon(X, Y) = \varepsilon B_\varepsilon(Y, X)$, for all $X, Y \in V$. If $\varepsilon = 1$ then the form B_1 is symmetric and we say V a *quadratic* vector space. If $\varepsilon = -1$ then m must be even and we say V a *symplectic* vector space with symplectic form B_{-1}. We denote by $\mathrm{End}(V)$ the *algebra of linear operators* of V and by

$\mathrm{GL}(V)$ the *group of invertible operators* in $\mathrm{End}(V)$. A map $C \in \mathrm{End}(V)$ is called *skew-symmetric* (with respect to B_ε) if it satisfies the following condition:

$$B_\varepsilon(C(X),Y) = -B_\varepsilon(X,C(Y)),\ \forall\, X,Y \in V.$$

We define $I_\varepsilon(V) = \{A \in \mathrm{GL}(V) \mid\ B_\varepsilon(A(X),A(Y)) = B_\varepsilon(X,Y),\ \forall\, X,Y \in V\}$ and

$$\mathfrak{g}_\varepsilon(V) = \{C \in \mathrm{End}(V) \mid\ C \text{ is skew-symmetric}\}.$$

Then $I_\varepsilon(V)$ is the *isometry group* of the bilinear form B_ε and $\mathfrak{g}_\varepsilon(V)$ is its Lie algebra. Denote by $A^* \in \mathrm{End}(V)$ the *adjoint map* of an element $A \in \mathrm{End}(V)$ with respect to B_ε, then $A \in I_\varepsilon(V)$ if and only if $A^{-1} = A^*$ and $C \in \mathfrak{g}_\varepsilon(V)$ if and only if $C^* = -C$. If $\varepsilon = 1$ then $I_\varepsilon(V)$ is denoted by $\mathrm{O}(V)$ and $\mathfrak{g}_\varepsilon(V)$ is denoted by $\mathfrak{o}(V)$. If $\varepsilon = -1$ then $\mathrm{Sp}(V)$ stands for $I_\varepsilon(V)$ and $\mathfrak{sp}(V)$ stands for $\mathfrak{g}_\varepsilon(V)$.

The *adjoint action* Ad of $I_\varepsilon(V)$ on $\mathfrak{g}_\varepsilon(V)$ is given by $\mathrm{Ad}_U(C) = UCU^{-1}$, for all $U \in I_\varepsilon(V)$, $C \in \mathfrak{g}_\varepsilon(V)$. We denote by $\mathscr{O}_C = \mathrm{Ad}_{I_\varepsilon(V)}(C)$, the *adjoint orbit* of an element $C \in \mathfrak{g}_\varepsilon(V)$ by this action.

If $V = \mathbb{C}^m$, we call B_ε a *canonical bilinear form* of $\mathbb{C}^m$. With respect to B_ε, we define a *canonical basis* $\mathscr{B} = \{E_1,\dots,E_m\}$ of $\mathbb{C}^m$ as follows. If m is even, $m = 2n$, write $\mathscr{B} = \{E_1,\dots,E_n,F_1,\dots,F_n\}$, if m is odd, $m = 2n+1$, write $\mathscr{B} = \{E_1,\dots,E_n,G,F_1,\dots,F_n\}$ and one has:

- if $m = 2n$ then for $1 \le i,j \le n$,

  $$\begin{cases} B_1(E_i,F_j) = B_1(F_j,E_i) = \delta_{ij},\ B_1(E_i,E_j) = B_1(F_i,F_j) = 0, \\ B_{-1}(E_i,F_j) = -B_{-1}(F_j,E_i) = \delta_{ij},\ B_{-1}(E_i,E_j) = B_{-1}(F_i,F_j) = 0 \end{cases}$$

  In the case of $\varepsilon = -1$, $\mathscr{B}$ is also called a *Darboux* basis of $\mathbb{C}^{2n}$.

- if $m = 2n+1$ then $\varepsilon = 1$ and for $1 \le i,j \le n$,

  $$\begin{cases} B_1(E_i,F_j) = \delta_{ij},\ B_1(E_i,E_j) = B_1(F_i,F_j) = 0, \\ B_1(E_i,G) = B_1(F_j,G) = 0, \\ B_1(G,G) = 1. \end{cases}$$

Also, in the case $V = \mathbb{C}^m$, we denote by $\mathrm{GL}(m)$ instead of $\mathrm{GL}(V)$, $\mathrm{O}(m)$ stands for $\mathrm{O}(V)$ and $\mathfrak{o}(m)$ stands for $\mathfrak{o}(V)$. If $m = 2n$ then $\mathrm{Sp}(2n)$ stands for $\mathrm{Sp}(V)$ and $\mathfrak{sp}(2n)$ stands for $\mathfrak{sp}(V)$. We will also write $I_\varepsilon = I_\varepsilon(\mathbb{C}^m)$ and $\mathfrak{g}_\varepsilon = \mathfrak{g}_\varepsilon(\mathbb{C}^m)$. Our goal is classifying all of I_ε-adjoint orbits of $\mathfrak{g}_\varepsilon$.

Finally, let V is an m-dimensional vector space. If V is quadratic then V is isometrically isomorphic to the quadratic space $(\mathbb{C}^m,B_1)$ and if V is symplectic then V is isometrically isomorphic to the symplectic space $(\mathbb{C}^m,B_{-1})$ [Bou59].

7.1. Nilpotent orbits

Recall the notation in Remark 4.12. The following Propoistion is widely known:

Proposition 7.1. *[Ger61] Nilpotent I_ε-adjoint orbits in $\mathfrak{g}_\varepsilon$ are in one-to-one correspondence with the set of partitions in $\mathscr{P}_\varepsilon(m)$.*

Next, we give a construction of a nilpotent element in $\mathfrak{g}_\varepsilon$ from a partition $[d]$ of m that is useful for this paper. Consider J_p the Jordan block of size $p \in \mathbb{N}^*$.

- For $p \geq 2$, we equip the vector space $\mathbb{C}^{2p}$ with its canonical bilinear form B_ε and the map C^J_{2p} having the matrix $C^J_{2p} = \begin{pmatrix} J_p & 0 \\ 0 & -{}^tJ_p \end{pmatrix}$ in a canonical basis where ${}^tJ_p$ denotes the *transpose* matrix of J_p. Then $C^J_{2p} \in \mathfrak{g}_\varepsilon(\mathbb{C}^{2p})$.
- For $p \geq 1$ we equip the vector space $\mathbb{C}^{2p+1}$ with its canonical bilinear form B_1 and the map C^J_{2p+1} having the matrix $C^J_{2p+1} = \begin{pmatrix} J_{p+1} & M \\ 0 & -{}^tJ_p \end{pmatrix}$ in a canonical basis where $M = (m_{ij})$ denotes the $(p+1) \times p$-matrix with $m_{p+1,p} = -1$ and $m_{ij} = 0$ otherwise. Then $C^J_{2p+1} \in \mathfrak{o}(2p+1)$.
- For $p \geq 1$, we consider the vector space $\mathbb{C}^{2p}$ equipped with its canonical bilinear form B_{-1} and the map C^J_{p+p} with matrix $\begin{pmatrix} J_p & M \\ 0 & -{}^tJ_p \end{pmatrix}$ in a canonical basis where $M = (m_{ij})$ denotes the $p \times p$-matrix with $m_{p,p} = 1$ and $m_{ij} = 0$ otherwise. Then $C^J_{p+p} \in \mathfrak{sp}(2p)$.

For each partition $[d] \in \mathscr{P}_{-1}(2n)$, $[d]$ can be reordered as

$$(p_1, p_1, p_2, p_2, \ldots, p_k, p_k, 2q_1, \ldots, 2q_\ell)$$

with all p_i odd, $p_1 \geq p_2 \geq \cdots \geq p_k$ and $q_1 \geq q_2 \geq \cdots \geq q_\ell$. We associate $[d]$ to the map $C_{[d]}$ with matrix $\mathrm{diag}_{k+\ell}(C^J_{2p_1}, C^J_{2p_2}, \ldots, C^J_{2p_k}, C^J_{q_1+q_1}, \ldots, C^J_{q_\ell+q_\ell})$ in a canonical basis of $\mathbb{C}^{2n}$ then $C_{[d]} \in \mathfrak{sp}(2n)$.

Similarly, let $[d] \in \mathscr{P}_1(m)$, $[d]$ can be reordered as

$$(p_1, p_1, p_2, p_2, \ldots, p_k, p_k, 2q_1+1, \ldots, 2q_\ell+1)$$

with all p_i even, $p_1 \geq p_2 \geq \cdots \geq p_k$ and $q_1 \geq q_2 \geq \cdots \geq q_\ell$. We associate $[d]$ to the map $C_{[d]}$ with matrix $\mathrm{diag}_{k+\ell}(C^J_{2p_1}, C^J_{2p_2}, \ldots, C^J_{2p_k}, C^J_{2q_1+1}, \ldots, C^J_{2q_\ell+1})$ in a canonical basis of $\mathbb{C}^m$ then $C_{[d]} \in \mathfrak{o}(m)$.

By Proposition 7.1, this construction induces a bijection that maps a partition $[d]$ in $\mathscr{P}_\varepsilon(m)$ to a nilpotent I_ε-adjoint orbit in $\mathfrak{g}_\varepsilon$ through $C_{[d]}$.

7.2. Semisimple orbits

We recall a well-known result [CM93]:

Proposition 7.2. *Let $\mathfrak{g}$ be a semisimple Lie algebra, $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$ and W be the associated Weyl group. Then there is a bijection between the set of semisimple orbits of $\mathfrak{g}$ and $\mathfrak{h}/W$.*

For each $\mathfrak{g}_\varepsilon$, we choose the Cartan subalgebra $\mathfrak{h}$ given by the vector space of diagonal matrices of type $\mathrm{diag}_{2n}(\lambda_1,\dots,\lambda_n,-\lambda_1,\dots,-\lambda_n)$ if $\mathfrak{g}_\varepsilon=\mathfrak{o}(2n)$ or $\mathfrak{g}_\varepsilon=\mathfrak{sp}(2n)$ and of type $\mathrm{diag}_{2n+1}(\lambda_1,\dots,\lambda_n,0,-\lambda_1,\dots,-\lambda_n)$ if $\mathfrak{g}_\varepsilon=\mathfrak{o}(2n+1)$.

Any semisimple $C\in\mathfrak{g}_\varepsilon$ is conjugate to an element of $\mathfrak{h}$. Denote by G_n the associated Weyl group of all coordinate permutations and sign changes of $(\lambda_1,\dots,\lambda_n)$.

Corollary 7.3. *Let $\Lambda_n=\{(\lambda_1,\dots,\lambda_n)\mid \lambda_1,\dots,\lambda_n\in\mathbb{C},\ \lambda_i\neq 0 \text{ for some } i\}$. Then there is a bijection between nonzero semisimple I_ε-adjoint orbits of $\mathfrak{g}_\varepsilon$ and Λ_n/G_n.*

7.3. Invertible orbits

Definition 7.4. We say I_ε-orbit $\mathscr{O}_X$ is *invertible* if X is an invertible element in $\mathfrak{g}_\varepsilon$.

Using the previous notation, we call $X\in V$ *isotropic* if $B_\varepsilon(X,X)=0$ and a subset $W\subset V$ *totally isotropic* if $B_\varepsilon(X,Y)=0$ for all $X,Y\in W$.

We will recall the classification method in [DPU12] as follows. First, we need:

Lemma 7.5. *Let V be an even-dimensional vector space with a nondegenerate bilinear form B_ε. Assume $V=V_+\oplus V_-$ with $V_\pm$ totally isotropic vector subspaces.*

1. *Let $N\in\mathrm{End}(V)$ such that $N(V_\pm)\subset V_\pm$. We define maps $N_\pm$ by $N_+|_{V_+}=N|_{V_+}$, $N_+|_{V_-}=0$, $N_-|_{V_-}=N|_{V_-}$ and $N_-|_{V_+}=0$. Then $N\in\mathfrak{g}_\varepsilon(V)$ if and only if $N_-=-N_+^*$ and, in this case, $N=N_+-N_+^*$.*
2. *Let $U_+\in\mathrm{End}(V)$, U_+ invertible, $U_+(V_+)=V_+$ and $U_+|_{V_-}=\mathrm{Id}_{V_-}$. We define $U\in\mathrm{End}(V)$ by $U|_{V_+}=U_+|_{V_+}$ and $U|_{V_-}=\left(U_+^{-1}\right)^*|_{V_-}$. Then $U\in I_\varepsilon(V)$.*
3. *Let $N'\in\mathfrak{g}_\varepsilon(V)$ satisfying the assumptions of (1). Define $N_\pm$ as in (1). Moreover, we assume that there exists $U_+\in\mathrm{End}(V_+)$, U_+ invertible such that $N'_+|_{V_+}=\left(U_+\,N_+\,U_+^{-1}\right)|_{V_+}$. We extend U_+ to V by $U_+|_{V_-}=\mathrm{Id}_{V_-}$ and define $U\in I_\varepsilon(V)$ as in (2). Then one has $N'=U\,N\,U^{-1}$.*

Let us now consider $C\in\mathfrak{g}_\varepsilon$, C invertible. Then, m must be even (obviously, it happens if $\varepsilon=-1$), $m=2n$. We decompose $C=S+N$ into semisimple and nilpotent parts, $S,N\in\mathfrak{g}_\varepsilon$ by its Jordan decomposition. It is clear that S is invertible and $\lambda\in\Lambda$ if and only if $-\lambda\in\Lambda$ where Λ is the spectrum of S. Also, $b(\lambda)=b(-\lambda)$, for all $\lambda\in\Lambda$ with multiplicity $b(\lambda)$. Since N and S commute, we have $N(V_{\pm\lambda})\subset V_{\pm\lambda}$ where V_λ is the eigenspace of S corresponding to $\lambda\in\Lambda$. Consider the direct sum $W(\lambda) = V_\lambda \oplus V_{-\lambda}$ and define the equivalence relation $\mathcal{R}$ on Λ by $\lambda \mathcal{R} \mu$ if and only if $\lambda = \pm\mu$. Then $\mathbb{C}^{2n} = \overset{\perp}{\bigoplus\limits_{\lambda \in \Lambda/\mathcal{R}}} W(\lambda)$ and each $(W(\lambda), B_\lambda)$ is a vector space with the nondegenerate form B_λ given by $B_\lambda = B_\varepsilon|_{W(\lambda)\times W(\lambda)}$.

Fix $\lambda \in \Lambda$. We write $W(\lambda) = V_+ \oplus V_-$ with $V_\pm = V_{\pm\lambda}$. Then, according to the notation in Lemma 7.5, define $N_{\pm\lambda} = N_\pm$. Since $N|_{V_-} = -N_\lambda^*$, it is easy to verify that the matrices of $N|_{V_+}$ and $N|_{V_-}$ have the same Jordan form. Let $(d_1(\lambda), \ldots, d_{r_\lambda}(\lambda))$ be the size of the Jordan blocks in the Jordan decomposition of $N|_{V_+}$. This does not depend on a possible choice between $N|_{V_+}$ or $N|_{V_-}$ since both maps have the same Jordan type.

Next, recall the notation and observations in Remark 4.12: let $\mathscr{I}(2n)$ be the set of invertible elements in $\mathfrak{g}_\varepsilon$ and $\widetilde{\mathscr{I}}(2n)$ be the set of I_ε-adjoint orbits of elements in $\mathscr{I}(2n)$. There is a map $i : \mathscr{I}(2n) \to \mathscr{J}_n$. A parametrization of $\widetilde{\mathscr{I}}(2n)$ can be obtained as follows. Its proof can be converted from the Lie algebra case, the Reader may refer to the proof of Proposition 7.10 in [DPU12].

Proposition 7.6. *The map $i : \mathscr{I}(2n) \to \mathscr{J}_n$ induces a bijection $\widetilde{i} : \widetilde{\mathscr{I}}(2n) \to \mathscr{J}_n$.*

7.4. Adjoint orbits in the general case

Let us now classify I_ε-adjoint orbits of $\mathfrak{g}_\varepsilon$ in the general case. Let C be an element in $\mathfrak{g}_\varepsilon$ and consider its Fitting decomposition $\mathbb{C}^m = V_N \oplus V_I$ where V_N and V_I are stable by C, $C_N = C|_{V_N}$ is nilpotent and $C_I = C|_{V_I}$ is invertible. Since C is skew-symmetric, $B_\varepsilon(C^k(V_N), V_I) = (-1)^k B_\varepsilon(V_N, C^k(V_I))$ for any k then one has $V_I = (V_N)^\perp$. Also, the restrictions $B_\varepsilon^N = B_\varepsilon|_{V_N\times V_N}$ and $B_\varepsilon^I = B_\varepsilon|_{V_I\times V_I}$ are nondegenerate. Clearly, $C_N \in \mathfrak{g}_\varepsilon(V_N)$ and $C_I \in \mathfrak{g}_\varepsilon(V_I)$. By Subsection 7.1 and Subsection 7.3, C_N is attached with a partition $[d] \in \mathscr{P}_\varepsilon(n)$ and C_I corresponds to a triple $T \in \mathscr{J}_\ell$ where $n = \dim(V_N)$, $2\ell = \dim(V_I)$. Let $\mathscr{D}(m)$ be the set of all pairs $([d], T)$ such that $[d] \in \mathscr{P}_\varepsilon(n)$ and $T \in \mathscr{J}_\ell$ satisfying $n + 2\ell = m$. By the preceding remarks, there exists a map $p : \mathfrak{g}_\varepsilon \to \mathscr{D}(m)$. Denote by $\mathscr{O}(\mathfrak{g}_\varepsilon)$ the set of I_ε-adjoint orbits of $\mathfrak{g}_\varepsilon$ then we obtain the classification of $\mathscr{O}(\mathfrak{g}_\varepsilon)$ as follows:

Proposition 7.7. *The map $p : \mathfrak{g}_\varepsilon \to \mathscr{D}(m)$ induces a bijection $\widetilde{p} : \mathscr{O}(\mathfrak{g}_\varepsilon) \to \mathscr{D}(m)$.*

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